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## ABSTRACT

Reported is research conducted as a part of the Project on Analysis of Mathematics Instruction. The study had two main purposes: to test the feasibility of teaching topics in probability and statistics to a class of sixth grade students; and to construct a set of instructional materials and procedures in probability and statistics for sixth graders. A unit of instruction was prepared and the order in which behavioral objectives were to be taught was determined from a content outline and a task analysis. The results of the study support the feasibility of teaching most of the topics covered in the unit to average and above average sixth graders. The study also lends support to the use of the systems model employed for developing curriculum materials. Part III is a continuation of Appendix A which was begun in Part II. Lesson Plans, exercises, and quizzes used in the study are included. (FL)

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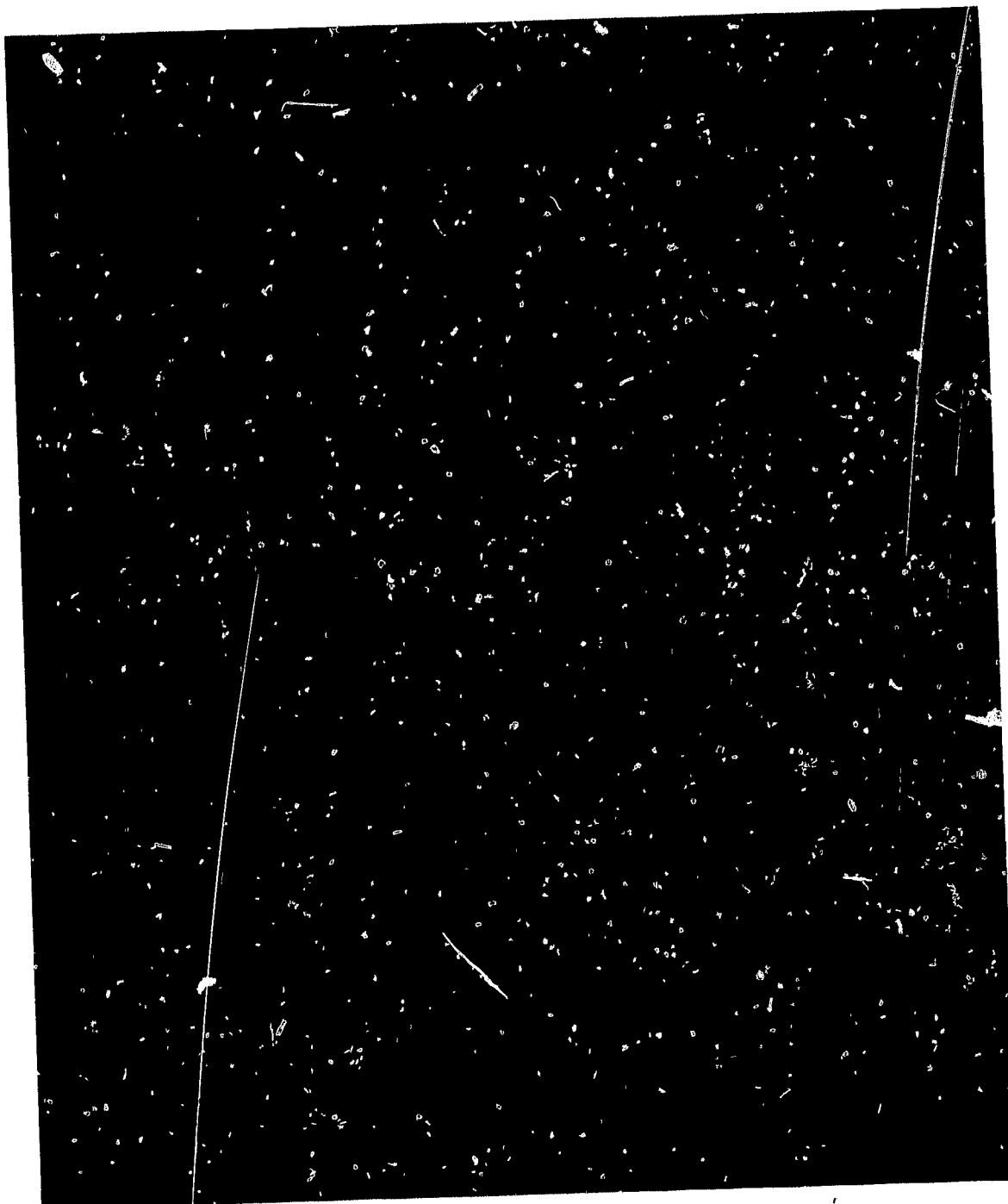
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No. 105 (Part III) Appendix A (Conclusion)

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## STATEMENT OF FOCUS

The Wisconsin Research and Development Center for Cognitive Learning focuses on contributing to a better understanding of cognitive learning by children and youth and to the improvement of related educational practices. The strategy for research and development is comprehensive. It includes basic research to learning and about the processes of instruction, and the subsequent development of research-based instructional materials, many of which are designed for use by teachers and others for use by students. These materials are tested and refined in school settings. Throughout these operations behavioral scientists, curriculum experts, academic scholars, and school people interact, insuring that the results of Center activities are based soundly on knowledge of subject matter and cognitive learning and that they are applied to the improvement of educational practice.

This Technical Report is from Phase 2 of the Project on Prototype Instructional Systems in Elementary Mathematics in Program 2. General objectives of the Program are to establish rationale and strategy for developing instructional systems, to identify sequences of concepts and cognitive skills, to develop assessment procedures for those concepts and skills, to identify or develop instructional materials associated with the concepts and cognitive skills, and to generate new knowledge about instructional procedures. Contributing to the Program objectives, the Mathematics Project, Phase 1, is developing and testing a televised course in arithmetic for Grades 1-6 which provides not only a complete program of instruction for the pupils but also inservice training for teachers. Phase 2 has a long-term goal of providing an individually guided instructional program in elementary mathematics. Preliminary activities include identifying instructional objectives, student activities, teacher activities materials, and assessment procedures for integration into a total mathematics curriculum. The third phase focuses on the development of a computer system for managing individually guided instruction in mathematics and on a later extension of the system's applicability.

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## ABSTRACT

From a content outline and a task analysis the behavioral objectives for a unit of instruction in probability and statistics for sixth-grade students and the order in which objectives would be taught were determined. An instructional analysis of the unit was undertaken to select or develop materials and procedures for teaching the unit.

Data from a pilot study conducted in the fall of 1969 were used to identify a set of nine lessons that could be formatively evaluated to test the feasibility of the instructional analysis. The lessons were used to teach a class of 25 sixth-grade students of average to above average ability. The topics developed through experiments, games and exercises were subjective probability notions, empirical probability, counting techniques, a priori probability including simple and compound events, and comparison of two events using probability.

On the basis of the overall pretest and posttest the instructional treatment was generally successful. The pretest percentage was 37.9% and the posttest percentage was 92.8% with all 72 items successful for 11 of the 14 measured objectives. Instruction was unsuccessful in getting students to specify the estimated probability; number the outcomes of an event; and estimate the probability successful for these three objectives because of a lack of stress and practice. Two learning hierarchies were also tested. One hierarchy was validated and the other was not. The results of the study support the feasibility of teaching most of the included topics in probability and statistics to average and above average sixth-grade students given high quality of teaching. The study lends support to the use of the systems developmental model employed in this study for developing curriculum materials for the schools, especially when used in conjunction with Bloom's "Mastery Learning" techniques.

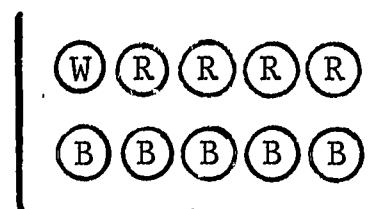
## QUIZ\*

Name \_\_\_\_\_

True (T) or False (F)

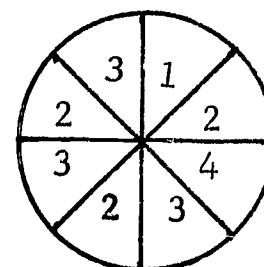
- \_\_\_\_\_ 1. When you spin a fair spinner 50 times that is  $\frac{1}{2}$  Red and  $\frac{1}{2}$  Blue you will most likely get between 20 and 30 Red.
- \_\_\_\_\_ 2. You have spun a fair spinner 50 times that is half Red and half Blue. You have gotten 30 Red and 20 Blue. If you were to spin the spinner again, you are more likely to get a Red than a Blue.
- \_\_\_\_\_ 3. It is possible for two teams to perform the experiment of spinning the spinner in Problem (2) 50 times and one team get 20 Red and 30 Blue and another team get 31 Red and 19 Blue.

You pick a marble from the box at the right containing red, white and blue marbles.



- \_\_\_\_\_ 1. How many possible outcomes are there?
- \_\_\_\_\_ 2. What is the probability of getting a blue marble?
- \_\_\_\_\_ 3. What is the probability of getting a red or a blue marble?
- \_\_\_\_\_ 4.  $P(R \text{ or } W \text{ or } B) = ?$

You spin the spinner at the right one time.  
(If it falls on a line you spin it again.)

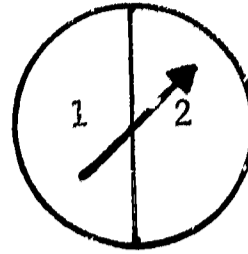


- \_\_\_\_\_ 1. How many possible outcomes are there? (Note: falling on a line is never considered as one of the possible outcomes since you spin again when it happens.)
- \_\_\_\_\_ 2.  $P(2) = ?$
- \_\_\_\_\_ 3.  $P(2 \text{ or } 4) = ?$
- \_\_\_\_\_ 4. What is the probability of getting a number greater than 7?

\_\_\_\_\_

\*Quiz IIA (first 14 items) and Quiz IIB (last 10 items)

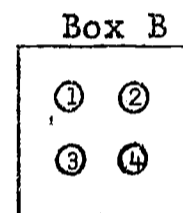
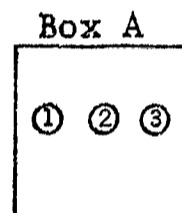
You spin the spinner at the right twice.  
The outcomes are:



1. 1,1
2. 1,2
3. 2,1
4. 2,2

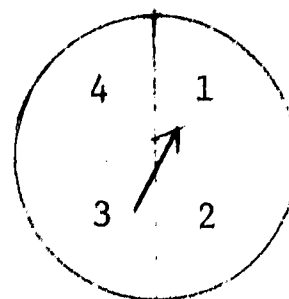
- \_\_\_\_\_ 1.  $P(1,1) = ?$
- \_\_\_\_\_ 2.  $P(1 \text{ on the first spin and any number on the second spin}) = ?$
- \_\_\_\_\_ 3.  $P(2 \text{ numbers are both even}) = ?$

For the following problems you pick a numbered chip from Box A and one from Box B.



- \_\_\_\_\_ 1. How many possible outcomes are there?
- \_\_\_\_\_ 2. What is the probability of getting "2" from Box A and "3" from Box B.
- \_\_\_\_\_ 3. What is the probability of getting an outcome whose sum is equal to 6?
- \_\_\_\_\_ 4. What is the probability of getting an outcome whose sum is less than 8?

For the following problems you spin the spinner at the right two times.



- \_\_\_\_\_ 1. How many possible outcomes are there?
- \_\_\_\_\_ 2. What is the probability of getting a "3" on the first spin and a "4" on the second spin?
- \_\_\_\_\_ 3. What is the probability of getting an outcome whose sum is equal to 8 ?
- \_\_\_\_\_ 4. What is the probability of getting an outcome whose sum is equal to 5 ?
- \_\_\_\_\_ 5. What is the probability of getting an outcome whose sum is greater than "10"?
- \_\_\_\_\_ 6. What is the probability of getting two even numbers?

## ANALYSIS OF QUIZ IIA AND QUIZ IIB

Thursday (3/20)

This quiz given on Thursday contained two subquizzes: Quiz IIA and Quiz IIB, one (first 14 items) to test their mastery of the concepts that were on the first quiz and the second (the last 10 items) to test their mastery of Lesson 6, Part I.

The mean on Quiz IIA in percentages was 90 per cent or 12.54 on 14 items with variance of 1.83. 18s/24s were classified as masters on the first part (misses  $\leq 2$ ). (Those missing 3 or 4 were subjects 9, 10, 15, 19, 20 and 25.) 15s/24/ missed 0 - 1; 21s/24s missed 0 - 3; 24s/24s missed 0 - 4. Three people, 10, 19, and 25, who were classified as masters on the first quiz were nonmasters on this quiz. In interviewing two subjects, 10 and 19 of these three, they said that they had looked at the problem at the top of page 2 and had answered the question by looking at the picture (i.e., treating the problem as a one-dimensional problem). From this conversation the author was convinced that they were masters of the concepts. This made 20s/24s masters at which point the author decided no more class quizzes would be given to measure the concepts that had been measured in Quiz I and Quiz IA. The item difficulties for the 14 items were:

1. 23/24	1. 23/24	1. 23/24	1. 19/24
2. 22/24	2. 24/24	2. 18/24	2. 19/24
3. 23/24	3. 23/24	3. 20/24	3. 18/24
	4. 24/24	4. 24/24	

In regard to Quiz II - B (10 items) only 6/24 were classified as Masters (90 per cent or better), 11/24 at 80 per cent or better, 14/24 at 70 per cent or better. The mean performance was 6.96 or 69.6 percent. The quiz was corrected such that if one missed the number of possible outcomes the rest of the items dependent on this were also marked wrong. This was done to emphasize the importance of counting correctly. If one scored the items so that for a question asking for the probability of an event is marked correct if a person used the answer he put for the number of possible outcomes in the denominator and the correct number in the numerator then the mean was 7.8 or 78 per cent.

Item difficulties for Quiz II - B:

<u>first way of scoring</u>	<u>second way of scoring</u>
1. 20/24	same
2. 18/24	21/24
3. 14/24	19/24
4. 23/24	same
1. 17/24	same
2. 16/24	21/24
3. 15/24	20/24
4. 13/24	15/24
5. 23/24	same
6. 8/24	9/24

The error in Item 1 was that the students added 3 and 4 together to get 7 (4/4 who missed the item).

The analysis of item 1 on the last page is not clear-cut. Most of the incorrect responses (5/7) viewed the problem as a one dimensional

problem and answered 4. The other two incorrect responses were 8 and 20.

Item 3, page 2, was missed by five people because of not being able to count the number of ways one can get the sum of 6. For Item 4, page 3, nine subjects had the same difficulty in finding the number of ways one gets the sum of 5. Item 6, page 3, was missed by 15 subjects, probably caused by people not viewing this as a two-dimensional counting problem, not counting correctly, or not knowing what is meant by two even numbers.

The class was told that they would have another chance to obtain mastery of the concepts contained in Quiz IIB.

Subjects scoring at the 60 per cent level or below were 10, 13, 16, and 25.

The quiz again showed that there should have been a worksheet in Lesson 6 which mixed one and two-dimensional problems in probability before this quiz. Although the author viewed the test as two quizzes, many subjects had problems in distinguishing between one and two-dimensional probability problems. This is understandable in the sense that the worksheets used in Lessons 4, 5 and 6 never mixed one-dimensional and two-dimensional problems together.

Subject 6 was absent for Quiz II.

## Lesson 7

## CHOOSING BETWEEN TWO EVENTS

Objectives:

The child should be able to:

1. Identify the most likely event of two unequally likely events in a one-dimensional sample space.
2. Identify the most likely event of two unequally likely events. (where one event is 1-D and the second event is 2-D).
3. Identify two equally likely events in a one-dimensional space as being equally likely.
4. Identify two equally likely events where one event is 1-D and the second event is in 2-D as being equally likely.

Prerequisite Behaviors:

The child should be able to:

1. Order two fractions (e.g.,  $3/7 > 2/6$ ).
2. Specify the probability of events in one and two dimensions.

Materials to be used:

1. Boxes and marbles.
2. Spinners.

New Vocabulary:

None.

Method of Presentation:

Play a game using two boxes of marbles, Box A containing 2 black and 4 white marbles and Box B containing 1 black and 1 white marble. A mark is placed on the board each time a black marble is drawn from Box A or Box B. The marks for Box A and Box B are recorded separately. Each person in the class is allowed to decide which box he would rather pick from. Thus two teams are formed. Team A using Box A and Team B using Box B. A person from Team A is to shake Box B so that members from Team B may take turns drawing. Do the same thing for Team A. Have one member from each team record the results on the board. Play the game drawing 30 times from each box. Team B should win.

Ask the teams why Team B won (if it does - and it should). If no one suggests probability ask them to assign a probability number to each box and compare the two fractions. Ask them on the basis of the probability numbers how many black they would expect in 30 draws. Compare this number with the actual number gotten.

Pass out handout concerning 1 D "urn" (box) problems where students are to decide which of two boxes they would rather draw from, or that it doesn't make any difference.

Ask them how they could use probability to make a decision as to which box they would rather pick from. Their reply should state that one should assign a probability to each box and compare the two fractions. If one is larger than the other then it is better to pick from the box with the larger fraction. If the two fractions are equivalent, it doesn't

make any difference.

Stress, "because of the need to compare two fractions I am going to pass out a sheet which asks you to compare 30 sets of fractions. These fractions are ones that will actually be used in future problems. If you know how to assign probabilities and compare two fractions you will have no trouble making the correct decision about the two boxes."

After the sheet of problems involving the ordering of two ratios is completed, correct them in class by having the children exchange their papers. Then ask them to do the rest of the problems concerning the 1 D box problems.

## Lesson 7, Part 1

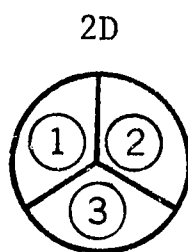
FRIDAY, MARCH 21

1. Pass out Exercise I (Lesson 7, Part 1) for students to finish.
2. Discuss Quiz.
3. Place the problems from the following sheet on the board.
4. Have students fill out ordering of two fractions exercise.
5. Introduce bonus problem concerning how many different telephone numbers there are.
6. Talk about chances of two people in the room having a birthday on the same day. Use picture from Time-Life's Mathematics concerning the problem.

## SUPPLEMENT TO LESSON 7

Class Exercise Arising from  
Analysis of Quiz II - B

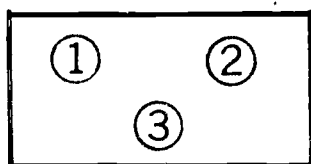
Friday (3/21)



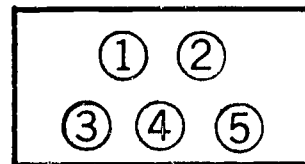
spin twice

1. How many possible outcomes?
2. How can we find out the number of possible outcomes?
3. What is the probability of 3 on the first spin, 3 on the second spin?
4. What is the probability of getting the sum of three?
5. What is the probability of getting two odd numbers?

Box A



Box B



1. What is the probability of getting 2 in Box B?
2. How many possible outcomes are there in Box B?
3. How many possible outcomes if you pick one chip from Box A and one chip from Box B?
4. What is the difference in picking from Box B only and picking from Box A and Box B?
5. What is the probability of getting the sum of 6?

## DISCUSSION OF LESSON 7, PART I (DAY TWO) AND

## REVIEW OF LESSON 6

Friday (3/21)

Seven minutes were spent to allow the students to finish the handout from Lesson 7, Part I.

This handout was then collected in order to correct it. The quizzes from Thursday were returned and discussed.

From the results of Quiz IIA, 20s/25s were classified as masters of probability as measured by Quiz I or Quiz IIA. However, students were confused by probability problems which called for using a tree as measured by Quiz IIB and only 11s/25s were at the 80 per cent level or above. Due to this, rather than start Lesson 7, Part II, which depends on this, it was decided to give students more practice in doing such problems and also help them to distinguish between one-dimensional and two-dimensional probability problems. The students were told that they would have a second chance to become masters on Monday. A quiz similar to Quiz IIB would be given at that time.

The following questions were written on the board and covered via class discussion. These took the rest of the period to complete.

If time had been available, the practice sheet for ordering two fractions would have been done as a class activity. (See pages

The following boxes contain black and white marbles. To play this game you pick a marble from one of the two boxes. You win if you choose a black marble.

If you can play this game only once, do you have a better chance of winning if you pick from Box A or Box B, or doesn't it make any difference?

For each question, place an "X" in the blank at the right that shows your choice.

1.
 

Box A
○ ○ ● ○

Box B
● ○ ○ ○ ○ ●

\_\_\_\_\_ Box A  
 \_\_\_\_\_ Box B  
 \_\_\_\_\_ It doesn't make any difference
  
2.
 

Box A
● ○

Box B
● ● ○ ○

\_\_\_\_\_ Box A  
 \_\_\_\_\_ Box B  
 \_\_\_\_\_ It doesn't make any difference
  
3.
 

Box A
○ ● ○ ○ ● ○ ○ ○ ○ ○

Box B
● ○ ○ ○ ● ○ ○ ● ○

\_\_\_\_\_ Box A  
 \_\_\_\_\_ Box B  
 \_\_\_\_\_ It doesn't make any difference
  
4.
 

Box A
○ ● ●

Box B
● ○ ○ ○ ● ●

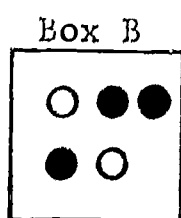
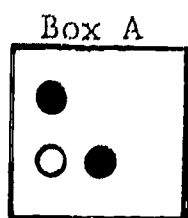
\_\_\_\_\_ Box A  
 \_\_\_\_\_ Box B  
 \_\_\_\_\_ It doesn't make any difference
  
5.
 

Box A
● ○

Box B
○ ● ● ○ ○ ○

\_\_\_\_\_ Box A  
 \_\_\_\_\_ Box B  
 \_\_\_\_\_ It doesn't make any difference

6.

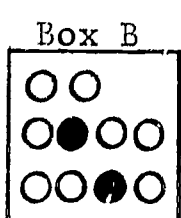
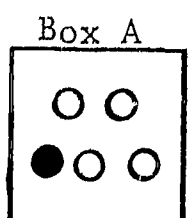


\_\_\_\_\_ Box A

\_\_\_\_\_ Box B

\_\_\_\_\_ It doesn't make any difference

7.

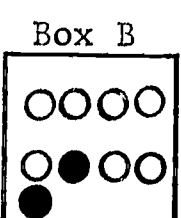
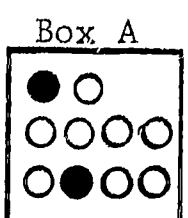


\_\_\_\_\_ Box A

\_\_\_\_\_ Box B

\_\_\_\_\_ It doesn't make any difference

8.

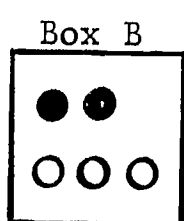
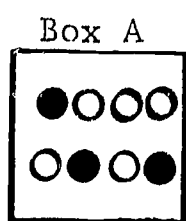


\_\_\_\_\_ Box A

\_\_\_\_\_ Box B

\_\_\_\_\_ It doesn't make any difference

9.

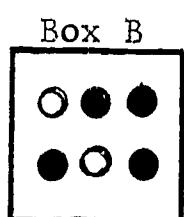
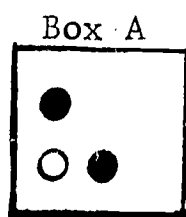


\_\_\_\_\_ Box A

\_\_\_\_\_ Box B

\_\_\_\_\_ It doesn't make any difference

10.

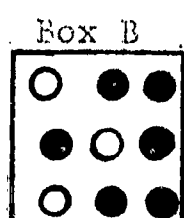
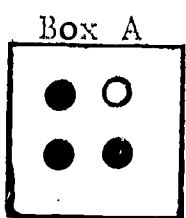


\_\_\_\_\_ Box A

\_\_\_\_\_ Box B

\_\_\_\_\_ It doesn't make any difference

11.

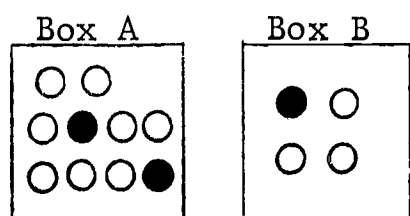


\_\_\_\_\_ Box A

\_\_\_\_\_ Box B

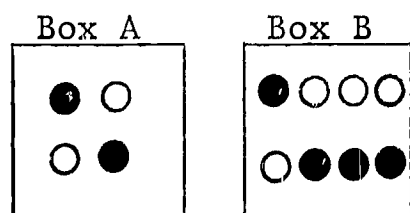
\_\_\_\_\_ It doesn't make any difference

12.



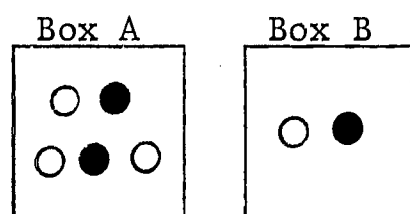
- \_\_\_\_\_ Box A
- \_\_\_\_\_ Box B
- \_\_\_\_\_ It doesn't make any difference

13.



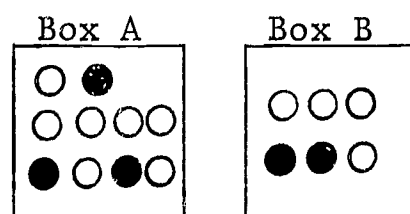
- \_\_\_\_\_ Box A
- \_\_\_\_\_ Box B
- \_\_\_\_\_ It doesn't make any difference

14.



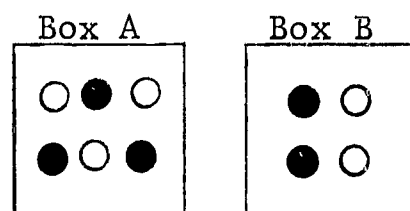
- \_\_\_\_\_ Box A
- \_\_\_\_\_ Box B
- \_\_\_\_\_ It doesn't make any difference

15.



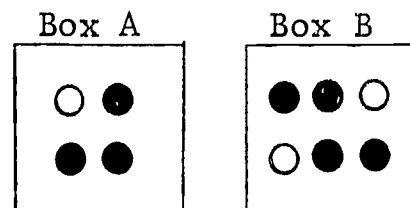
- \_\_\_\_\_ Box A
- \_\_\_\_\_ Box B
- \_\_\_\_\_ It doesn't make any difference

16.



- \_\_\_\_\_ Box A
- \_\_\_\_\_ Box B
- \_\_\_\_\_ It doesn't make any difference

17.



- \_\_\_\_\_ Box A
- \_\_\_\_\_ Box B
- \_\_\_\_\_ It doesn't make any difference

## DISCUSSION OF EXERCISE 7.1

Exercise 7.1 was done quite well. Other than for two subjects, 15 (score 5/17) and 20 (score 3/17), the lowest score was 13/17. 21/25 were classified as masters (missed  $\leq 2$ ). The overall mean was 87.1 per cent or 14.80 with a variance of 11.44. The mean percentage, leaving the subjects mentioned above out, was 93.2 per cent (based on 23 subjects).

Because of the performance on this exercise and the pre-test with ratios, it was decided not to give the class the practice sheet in ordering fractions. Individuals who need practice such as subjects 15 and 20 were given the practice sheet. The mistakes of the two subjects showed that in the majority of the items their decisions seem to be made in terms of which box had the most winners, without even considering the number of losers.

ITEM DIFFICULTY - EXERCISE 7.1  
Choosing Between Two Boxes (1D)

1	24/25	10	19/25
2	22/25	11	20/25
3	25/25	12	21/25
4	20/25	13	24/25
5	23/25	14	22/25
6	20/25	15	22/25
7	20/25	16	23/25
8	23/25	17	20/25
9	21/25		

## Lesson 7, Part II

MONDAY, MARCH 24

1. Review one-dimensional and two-dimensional probability problems by placing the following problems on the board.
2. Give Quiz (write on board).
3. Pass our Exercise II (Lesson 7, Part II).
4. For masters who complete exercise, have them start experiments.

DISCUSSION OF REVIEW OF LESSONS 4 AND 6  
AND OF THE INTRODUCTION TO LESSON 7, PART II

Monday (3/24)

On Monday the review problems involving one-dimensional and two-dimensional problems were put on the board for the class to practice on. The two-dimensional problems included sampling with and without replacement. The review lasted approximately 15 minutes.

The quiz was written on the board and lasted 15 minutes.

When most students had completed the quiz, the author put the Brain-teaser on the board of the number of different telephone numbers there are (using seven digits). The students were told that the problem was an extra credit problem which they could work on in their free time. The introduction of the brain teaser took five minutes.

The handout for Lesson 7, Part II was distributed. The first two questions were done together. The teacher emphasized the importance of assigning the probability numbers and then ordering these in order to make the best decision. In the second problem one girl was confused by  $1/5$ ,  $1/6$  and which was larger. The teacher explained to them that if the numerators remains the same and the denominator is made larger, the fraction gets smaller.

There seemed to be a semantic problem between the teacher and the girl. The girl could identify the larger fraction but said that you pick the smaller number (probably meaning the smaller denominator).

The discussion of the two problems and how to order two fractions took the last ten minutes of the period.

The handout was collected although only the two problems that were done together had been completed.

MONDAY - 3/24

SUPPLEMENT B TO LESSON 7

Review Lesson (Probability in two dimension)

Materials: Box and 3 plastic animals

Box and 5 numbered plastic chips (1, 2, 3, 4, 5)

1. Review probability in 2 dimensional by putting the following problems on the board.

(Show the class the box with the three plastic animals in it.)

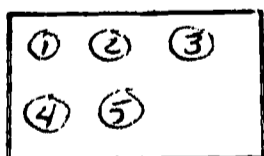
The following box has a plastic dog, cat, and cow.

- (1) You pick once from the box.  $P(\text{dog}) = ?$

You pick twice from the box and you put the first one you pick back.

(Note: Do the problems again only this time you do not put the first animal back.)

- (1) How many possible outcomes are there?
- (2) What is the probability of getting a cat on the first pick and a dog on the second pick?
- (3)  $P(\text{cat on the first pick and any animal on the second pick})$ .



You pick twice from the box and you put the first one you pick back.

(Note: Do the problems again only this time without putting the first chip back.)

- (1) What is the probability of getting ② on the first pick and ③ on the second pick?
- (2) What is the probability of getting a sum of 7?
- (3)  $P(\text{sum} = 10) = ?$

The problems will also introduce the problem of assigning a probability number to picking from a box twice with and without replacement (where the objects in the box are distinct).

2. Place the quiz on probability in two dimensional on the board.

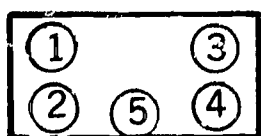
3. In what time remains pass out the exercise for Lesson 7 - II.

Do the first two problems together. Collect the exercise. (It will be completed in class tomorrow.)

## QUIZ III

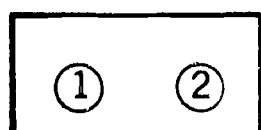
[Comment:

This quiz was written on the blackboard.]



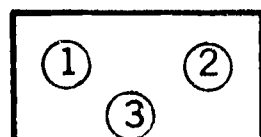
You pick one chip from the box.

1. Find the probability of getting a 3 or a 4.

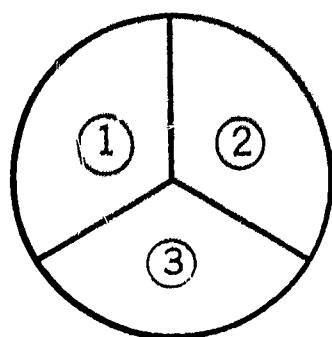


You pick one chip from Box A and one chip from Box B.

2. What is the probability of getting 1 from Box A and 3 from Box B?



3. What is the probability of getting a sum of 3?



You spin the spinner twice.

4. What is the probability of 1 on the first spin and 2 on the second spin?

5.  $P(\text{sum} = 4) = ?$

## ANALYSIS OF QUIZ III

Quiz III (five items written on the board) was concerned with one and two-dimensional probability problems. 13/23 got 5/5 questions correct and were classified as masters. The mean percentage was 80 per cent or 4.00 with a variance of 1.83. 16/23 got 4/5 or better; 19/23 got 3/5 or better. Those missing two or more problems were subjects 1, 2, 7, 13, 15 and 20.

One person, subject 4, who had been a master on Quiz IIB missed one on this. She also was the last one done with this quiz (she sat in the back of the class and said she had to rush because she had a hard time reading the board). On this basis and because of her other quiz, the experimenter decided to count her as a master. Thus, using this Quiz or Quiz IIB, 15/25 were classified as masters. (One boy who was absent, subject 17, had been a master on Quiz IIB.)

It was decided on the basis of these results that on Tuesday the class would be split into two groups after the introduction of Lesson 7, Part II. All masters would work on handout 7.2 while the nonmasters, as measured by these two quizzes, would be given further practice and drill by the author concerning concepts or probability problems in one-dimensional and two dimensional.

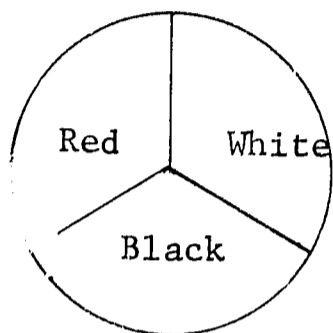
Item difficulties, Quiz III:

- |          |          |          |
|----------|----------|----------|
| 1. 19/23 | 3. 18/23 | 5. 18/23 |
| 2. 19/23 | 4. 18/23 |          |

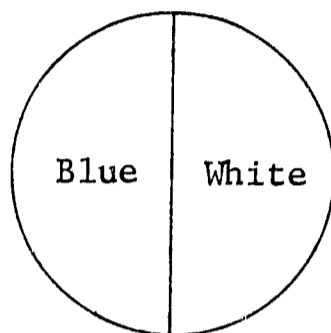
The following set of problems ask you to choose the game which gives you the best chance of winning. If the chances are the same you are to identify that it doesn't make any difference. Place an X in the blank to the left of the choice which you think is correct. Give a short explanation as to why you gave the answer you did. If you do not know the answer to a problem, leave it blank. Do not guess.

1. Which game would you rather play or doesn't it make any difference?

First Spinner



Second Spinner

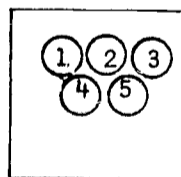


- \_\_\_\_\_ Game 1    You spin the first spinner once and you win if you get red.
- \_\_\_\_\_ Game 2    You spin both spinners once and you win if you get white and yellow or white and blue.
- \_\_\_\_\_ It doesn't make any difference.

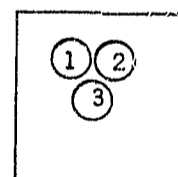
Why?

2. Which game would you rather play or doesn't it make any difference?

Box A



Box B

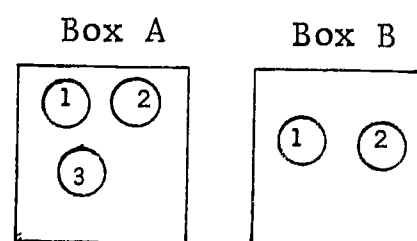


- \_\_\_\_\_ Game 1    You pick a chip from box A and you win if you pick the chip with "3" on it.
- \_\_\_\_\_ Game 2    You pick twice from Box B (without putting the first chip back). You win if you get a "1" on the first pick and a "3" on the second pick.

\_\_\_\_\_ It doesn't make any difference.

Why?

3. Which game would you rather play or doesn't it make any difference?



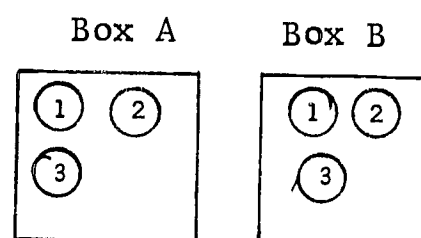
\_\_\_\_\_ Game 1     You pick a chip from Box A and you win if you pick the chip with "2" on it.

\_\_\_\_\_ Game 2     You pick one chip from Box A and one chip from Box B.  
You win if you get a "1" from Box A and a "2" from Box B.

\_\_\_\_\_ It doesn't make any difference.

Why?

4. Which game would you rather play or doesn't it make any difference?



\_\_\_\_\_ Game 1     You pick a chip from Box A and you win if you pick the chip with "2" on it.

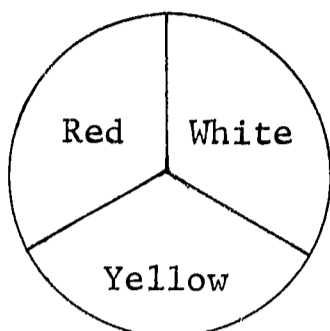
\_\_\_\_\_ Game 2     You pick one chip from Box A and one chip from Box B and find the sum of the numbers on the chips. You win if the sum is "3".

\_\_\_\_\_ It doesn't make any difference.

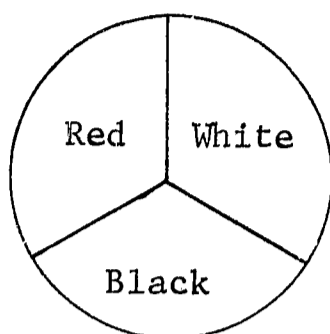
Why?

5. Which game would you rather play or doesn't it make any difference?

Spinner A



Spinner B



\_\_\_\_\_ Game 1     You spin Spinner A once and you win if you get white.

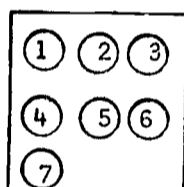
\_\_\_\_\_ Game 2     You spin Spinner B twice. You win if you get Red on the first spin and White on the second spin.

\_\_\_\_\_ It doesn't make any difference.

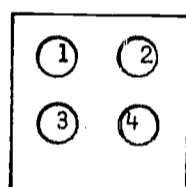
Why?

6. Which game would you rather play or doesn't it make any difference?

Box A



Box B



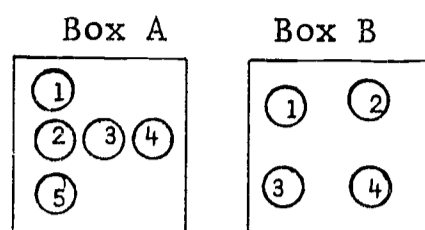
\_\_\_\_\_ Game 1     You pick a chip from Box A and you win if you pick the chip with "3" on it.

\_\_\_\_\_ Game 2     You pick twice from Box B without putting the first chip you pick back. You win if you pick a "2" first and a "3" on the second pick.

\_\_\_\_\_ It doesn't make any difference.

Why?

7. Which game would you rather play or doesn't it make any difference?



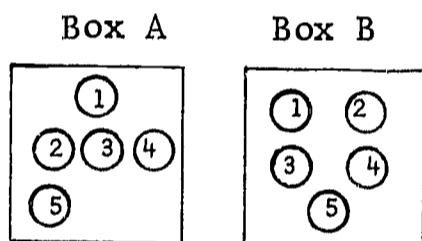
\_\_\_\_\_ Game 1    You pick one chip from Box A and you win if you get a chip with a "2" on it.

\_\_\_\_\_ Game 2    You pick one chip from Box A and one from Box B and find the sum of the numbers on the chips. You win if you get a sum of "3" or "6".

\_\_\_\_\_ It doesn't make any difference.

Why?

8. Which game would you rather play or doesn't it make any difference?



\_\_\_\_\_ Game 1    You pick one chip from Box A and you win if you get a chip with a "2" on it.

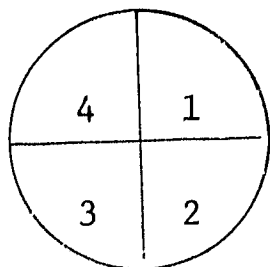
\_\_\_\_\_ Game 2    You pick two chips from Box B and find the sum of the numbers on the chips. You win if you get a sum of "5" or "9".

\_\_\_\_\_ It doesn't make any difference.

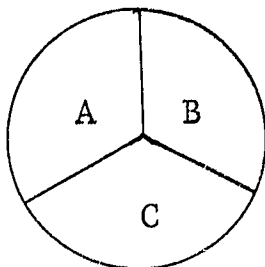
Why?

9. Which game would you rather play or doesn't it make any difference?

First Spinner



Second Spinner



\_\_\_\_\_ Game 1     You spin the first spinner once and you win if you get a "4".

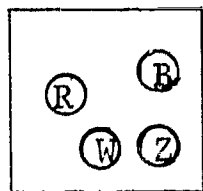
\_\_\_\_\_ Game 2     You spin both spinners once and you win if you get "1" and an "A" or a "1" and a "B".

\_\_\_\_\_ It doesn't make any difference.

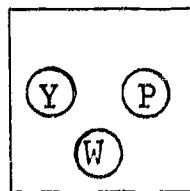
Why?

10. Which game would you rather play or doesn't it make any difference?

Box A



Box B



\_\_\_\_\_ Game 1     You pick a chip from Box A and you win if you get the chip with "R" on it.

\_\_\_\_\_ Game 2     You pick a chip from each box and you win if you get "R" and "Y" or "B" and "Y".

\_\_\_\_\_ It doesn't make any difference.

Why?

## Lesson 7, Part II

TUESDAY, MARCH 25

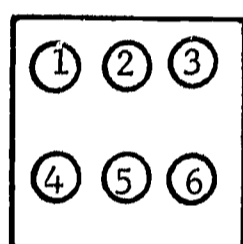
1. Do Lesson 7, Part II.
2. Pass back Exercise II (Lesson 7, Part II) to all students.
  - a. Have masters of probability work independently.
  - b. Have nonmasters work with the observer.
  - c. The teacher is to work with child who has been absent and answer questions of masters.
3. For masters who complete exercise, have them start experiments.

## Lesson 7, Part II

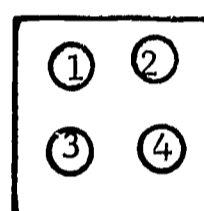
## ONE-DIMENSIONAL AND TWO-DIMENSIONAL GAMES

Begin the lesson by playing a game with children using the same procedure as in Lesson 7, Part 1. Only this time, the children will have to decide between the following two games. The game should be played 24 or 30 times.

Box A



Box B



Game 1. You pick one chip from Box A and you win if you pick the chip with a "3" or a "4" on it.

Game 2. You pick two times from Box B without putting the first chip back. You win if you get a sum of "3" when the two chips are added together.

Game 1 should win ( $2/6$  is twice as great as  $2/12$ ). Ask them if they can explain why. (If they do not, mention probability.)

Pass out handout involving 1 D and 2 D games. Work the first two problems together. The students should be able to work the rest of the problems alone.

## DISCUSSION OF LESSON 7, PART II

Tuesday (3/25)

The teacher drew a picture of each box on the board and also wrote the directions for the two games, as in the lesson plan. She asked the children to choose which game they would rather play and write it on a piece of paper.

The game was played with each child playing both games once. Two children passed the boxes and two children tallied the number of winners and losers for each game.

After the children had played the game, the teacher asked for the probability of winning each game. They said  $2/6$  and  $2/12$  respectively. The teacher asked them to tell her how they counted the outcomes. They said that in game 1 you just count, but you need to draw a tree for game 2. The teacher then verified the students' probabilities by counting the outcomes for each game. The teacher asked, "If the probability of winning in game 1 is  $2/6$  or  $1/3$  and we played the game 25 times, how many winners would you expect?" A student said eight. (This also was the number actually gotten when the game was played 25 times.)

Similar questions were asked for game 2. They replied "Four." (Again this was what was actually gotten.)

The teacher pointed out that the results didn't necessarily have to come out to that figure exactly, but the chances were greater that the results would come out near four. The game activity took 10 minutes.

The teacher then went over the entire goal chart and reviewed the probability of the certain even is equal to one while the probability of the impossible event is equal to 0. The teacher pointed out the particular goals that today's lesson was concerned with.

At this point the author told the class the results of the first quiz on probability problems needing a tree had produced 6/25 masters. On the second quiz Monday, 16/25 masters. In order to give the other nine the opportunity to become masters of these concepts, the class would be split up into two groups. The author would work with the children who were nonmasters while the rest of the class would work on the exercise for Lesson 7, Part II. When the latter group finished the exercise, they were told that they could work on experiments 1-4 and begin to collect data.

The teacher worked individually with subject 6 who had been absent for a week on the lessons he had missed. The author worked with subjects 1, 2, 5, 7, 13, 15, 20 and 25.

The author talked about games of chance, how people play games of chance in Las Vegas and how probability helps a person to make the best decisions in game situations.

The eight students were then presented counting and probability problems in a group situation involving picking from a box once, twice, once from Box A and once from Box B, spinning a spinner once, twice,  $P(A \text{ or } B)$  and  $P(A \text{ and } B)$  statements, how to use a tree; and the difference between

problems for which a tree was appropriate in contrast to those for which it was not.

The students were instructed that they could become masters if they assigned the correct probabilities to the games in the exercise of Lesson 7, Part II, problems 3-6. The students then worked on these problems individually. The author graded these when they were through and gave the papers back to them so that they could complete the rest of the exercise for tomorrow. Of the eight, three were classified as masters, (subjects 1, 2 and 7). Only two subjects were still confused (15 and 20). The other three made some inconsistent responses and probably would improve with more practice.

The exercises for Lesson 7, Part II of the students who were classified as masters and who had been working independently were collected at the end of the period.

Upon analyzing these papers it became clear that subjects 10 and 25 were having problems ordering fractions. Subject 25 would normally assign the correct probability number, but would always choose the game with the smaller fraction. Subject 10 was also having problems in assigning the correct probabilities. Due to this, it was decided to give the two girls the practice sheets on ordering two fractions and to include subject 10 in the extra help session on Wednesday.

Six children (6, 9, 10, 13, 15, 20 and 25) were included in Wednesday's extra help session. The session concentrated on the objectives of Lesson 6 and using probabilities to make decisions between two games (Lesson 7). The exercise from Lesson 7, Part II was a major source of problems.

## Lesson 8

WEDNESDAY, MARCH 26

1. Collect homework.
2. "Yesterday we assigned a probability number to games and compared the fractions in order to decide which game would give you a better chance of winning, or whether it didn't make any difference. Can you think of another way besides assigning probability numbers to make a good decision between two games?"  
  
Summarize: "Today we are going to carry out some experiments to see whether the probability number actually helped us make the correct decision."- 3. Explain the six experiments.
- 4. Explain the Data Sheets.
- 5. Form committees.
- 6. Pass out instructions and materials.
- 7. Pass out Sheet A at the end of the period and discuss. Show students models to be used.

## Lesson 8

## EXPERIMENTS--MOST (EQUALLY) LIKELY EVENTS

Objectives:

The child should be able to:

1. Perform the given experiment.
2. Specify the cumulative frequency (total number of occurrences) of an event from the data gathered.
3. Specify the estimated probability of an event from the data gathered in an experiment.
4. Identify the order of an estimated probability in comparison to an a priori probability.

Prerequisite Behaviors:

1. Collect and organize data.
2. Specify the probability of an event.

Materials to be used:

1. Marbles, boxes.
2. Tacks, cups (paper).
3. Spinners.
4. Handout.

New Vocabulary:

None.

Methods of Presentation:

If necessary, review graphing using cumulative data on the overhead. Present students with the following experiments. Ask students to state on the handout which of the two events in the experiment is more likely to happen and to specify the a priori probabilities. Enter into a discussion about how one could gather evidence to support or reject this decision. Help them to define and carry out an experiment, perhaps in groups of four so that two people could be working on each event and then exchange events half way through the experiment. Ask committees to record data when finished on a central data sheet and to sign their names. Since the committees will not be given organized data sheets for each experiment, they will have to decide how they will gather and record the data. Remind them of fair and unfair sampling procedures and of the importance of using fair sampling procedures since the data from the different committees will be compared and totaled together.

After a committee has performed one experiment, it should go on to another.

Show how one computes the estimated probability as the number of successes the number of times an experiment is performed in order that comparisons can be made between the a priori probability and the former. Ask them to record this also on the central data sheet. Have a committee take cumulative results from the different committees and be responsible for reporting, graphing and interpreting the results for a particular experiment.

Pass out the following sheet at the end of the lesson. Discuss content of the sheet and show students models to be used in the experiments.

## SHEET A

In the coming lessons we will need data gotten from the following experiments to help us understand probability better.

In order for us to collect the data we would like you to help by choosing any of the available models from the box in the corner. The models may be taken home overnight. Please sign out the model you take on the sheet above the box.

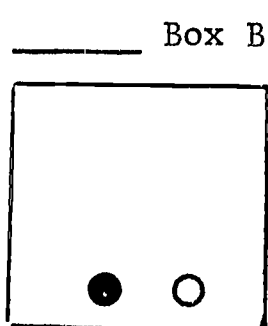
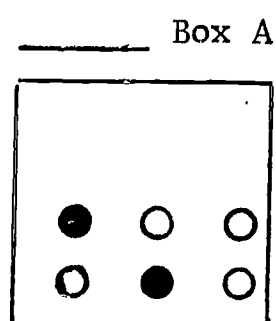
Each trial is to be done 50 times. You may do as many trials as you like. Please record your data on the Central Data Sheet for your experiments.

The experiments are:

1. To spin the oddly divided, red, & black spinner one time and record the number of red & black.
2. To spin the first spinner once, and the second spinner once.  
( $\frac{1}{2}$  R,  $\frac{1}{2}$  B) and ( $\frac{1}{3}$  R,  $\frac{1}{3}$  B,  $\frac{1}{3}$  Y) Record two things:  
the number of R R (Red, Red), and all other outcomes.
3. To toss the tack on a hard surface and record the number of times it turns up, and the number of times it is on its side. (Allow the tack to bounce to make it a fair toss).
4. To toss the paper cup on a hard surface and record the number of times it turns "up" the number of times it turns "down" and the number of times it is on its side. (Allow the cup to bounce to make it a fair toss).

## EXPERIMENT 1\*

The following two boxes contain black and white marbles. To play this game you pick a marble from one of the two boxes. You win if you choose a black marble. Which box would you rather pick from or doesn't it make any difference?



\_\_\_\_\_ it doesn't  
make any  
difference

Carry out an experiment to see whether the data you gather support your answer. Divide your committee into two teams. Each team is to take one box and pick a marble from the box without looking. One member of each team is to carry out the experiment while the other member acts as a recorder.

The experiment should be done 50 times for each box. The two teams may change boxes half way through the experiment if they wish.

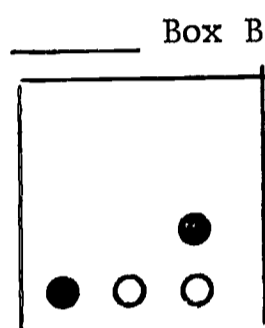
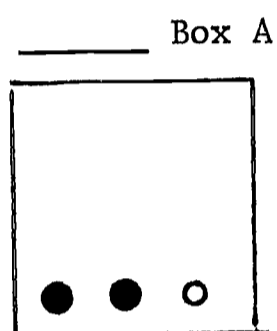
When you have finished the experiment, record your results on the Central Data Sheet for Experiment 1.

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\*Description of experiments given to students (Lesson 8).

## EXPERIMENT 2

The following two boxes contain black and white marbles. To play this game you pick a marble from one of the two boxes. You win if you choose a black marble. Which box would you rather pick from or doesn't it make any difference.



it doesn't  
make any  
difference

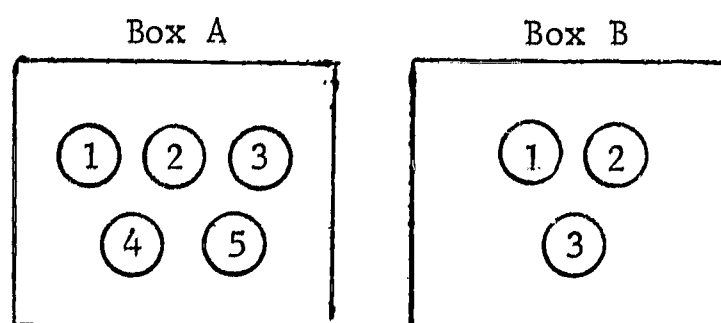
Carry out an experiment to see whether the data you gather support your answer. Divide your committee into two teams. Each team is to take one box and pick a marble from the box without looking. One member of each team is to carry out the experiment while the other member acts as a recorder.

The experiment should be done 50 times for each box. The two teams may change boxes half way through the experiment if they wish.

When you have finished the experiment, record your results on the Central Data Sheet for Experiment 2.

## EXPERIMENT 3

Which game would you rather play or doesn't it make any difference?



\_\_\_\_\_ Game 1      You pick a chip from Box A and you win if you pick the chip with a "3" on it.

\_\_\_\_\_ Game 2      You pick twice from Box B (without putting the first chip back). You win if you get a "1" on the first pick and a "3" on the second pick.

\_\_\_\_\_ It doesn't make any difference.

Carry out an experiment to see whether the data you gather support your answer.

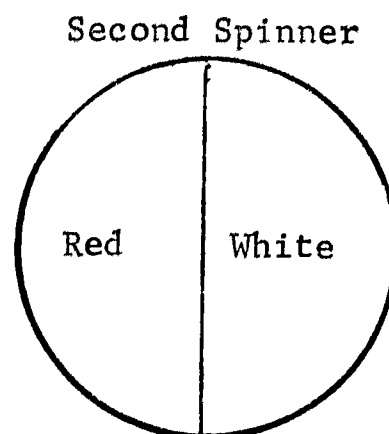
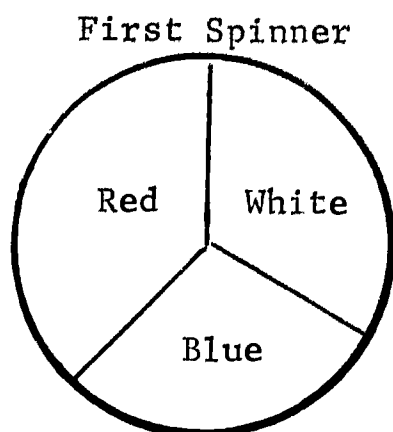
Divide your committee into two teams. One team is to play Game 1 and the other team is to play Game 2. One member of each team is to carry out the experiment while the other member acts as a recorder.

The experiment should be done 50 times for each game. The two teams may exchange games half way through the experiment if they wish.

When you have finished the experiment, record your results on the Central Data Sheet for Experiment 3.

## EXPERIMENT 4

Which game would you rather play or doesn't it make any difference?



\_\_\_\_\_ Game 1     You spin the first spinner once and you win if you get red.

\_\_\_\_\_ Game 2     You spin both spinners once and you win if you get white  
                                 on the first spinner and red on the second spinner.

\_\_\_\_\_ It doesn't make any difference.

Carry out an experiment to see whether the data you gather support your answer.


Divide your committee into two teams. One team is to play Game 1 and the other team is to play Game 2. One member of each team is to carry out the experiment while the other member acts as a recorder.


The experiment should be done 50 times for each game. The two teams may exchange games half way through the experiment if they wish.

When you have finished the experiment, record your results on the Central Data Sheet for Experiment 4.

## EXPERIMENT 5

Which game would you rather play or doesn't it make any difference?

\_\_\_\_\_ Game 1      You toss a tack and you win if it turns up 

\_\_\_\_\_ Game 2      You toss a cup and you win if it turns up 

\_\_\_\_\_ It doesn't make any difference.

Carry out an experiment to see whether the data you gather support your answer.

Divide your committee into two teams. One team is to play Game 1 and the other team is to play Game 2. When the objects are tossed into the air they should be allowed to bounce on a surface before coming to rest.

(We hope this will take all prejudice out of the way you toss. One member of each team is to carry out the experiment while the other member acts as a recorder.

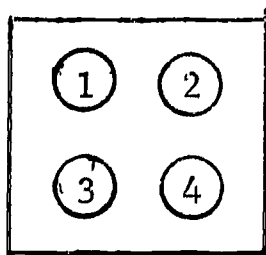
The experiment should be done 50 times for each game. The two teams may exchange games half way through the experiment if they wish.

When you have finished the experiment, record your results on the Central Data Sheet for Experiment 5.

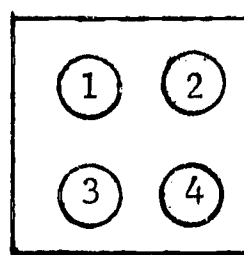
## EXPERIMENT 6

Which game gives you the best chance of winning or doesn't it make any difference?

Box A



Box B



\_\_\_\_\_ Game 1     You pick one chip from Box A and you win if you get a chip with a "4" on it.

\_\_\_\_\_ Game 2     You pick one chip from Box A and one from Box B and find the sum of the numbers on the chips. You win if you get a sum of 4 or 5.

\_\_\_\_\_ It doesn't make any difference.

Carry out an experiment to see whether the data you gather support your answer.

Divide your committee into two teams. One team is to play Game 1 and the other team is to play Game 2. One member of each team is to carry out the experiment while the other member acts as a recorder.

The experiment should be done 50 times for each game. The two teams may exchange games half way through the experiment if they wish.

When you have finished the experiment, record your results on the Central Data Sheet for Experiment 6.

## EXPERIMENT

318

Trials	Initials	No. of ____	50 in a Trial	Estimated Probability	Additive Total of ____	Additive Totals of Trials	Estimated Probability of Additive Total
T1							
T2							
T3							
T4							
T5							
T6							
T7							
T8							
T9							
T10							
T11							
T12							
T13							
T14							
T15							
T16							
T17							
T18							
T19							
T20							
T21							
T22							
T23							
T24							

## DATA SHEET

Name \_\_\_\_\_

Experiment \_\_\_\_\_

(Make sure you have a total of 50 for each trial)

Trial 1 (50 Times)

Number of \_\_\_\_\_

Number of \_\_\_\_\_

Number of \_\_\_\_\_

1			
2			
3			
4			
5			
Total			

Trial 2 (50 Times)

Number of \_\_\_\_\_

Number of \_\_\_\_\_

Number of \_\_\_\_\_

1			
2			
3			
4			
5			
Total			

Trial 3 (50 Times)

Number of \_\_\_\_\_

Number of \_\_\_\_\_

Number of \_\_\_\_\_

1			
2			
3			
4			
5			
Total			

Trial 4 (50 Times)

Number of \_\_\_\_\_

Number of \_\_\_\_\_

Number of \_\_\_\_\_

1			
2			
3			
4			
5			
Total			

## DISCUSSION OF LESSON 8

Wednesday (3/26)

First the teacher collected the homework. She then showed the children a cup and a tack and asked them which of these they would rather toss if they win when the tack or the cup lands up. The teacher told the students who had done the experiment with the cup and the tack not to answer. Those children who raised their hands and responded said the tack. Two children thought that it wouldn't make any difference. Several children said they weren't sure and wanted the teacher to tell them how many outcomes there were. She asked, "What could we do to find out which would give you a better chance of winning?" One girl responded, "We could do an experiment."

The teacher then explained each of the six experiments and showed the children the models to be used. She explained to them how to record the data. When the teacher asked the children if they understood, the students were atypically unresponsive. At this point, the author re-explained, trying to emphasize and convey to them why the activities were being done.

The class was then divided into committees of four with the teacher assigning each committee to its first experiment. The children were to choose the experiments they wished to do from the available materials thereafter.

The students became caught up in doing the activities and many did not work as a committee assigning the probabilities first, collecting

the data for each games in teams of two and then comparing their results. Part of this difficulty was caused by the students trying to complete as many activities as possible.

The rest of the period was spent in doing these activities.

## COMMENT ON LESSON 8

The sheets which explained the experiments were poorly constructed. The students focused on doing the experiment rather than on analyzing the situation and the data.

The following questions should have been on the sheet for Experiment 1.

1. What is the  $P(\text{winning for Box A})$ ?
2. What is the  $P(\text{winning for Box B})$ ?
3. Before carrying out the experiment, which box would most likely make you a winner?

Answer the following questions after carrying out the experiment 50 times.

4. How many winners did you get in Box A?
5. How many winners did you get in Box B?
6. Did the box that you expected to win have the most winners?
7. Were the numbers of winners close to the expected number of winners?

The other sheets describing the experiments should be modified in the same way. The investigator hopes that by making this change that students will become more involved in the process of analyzing a problem a priori, collecting data, and analyzing the data to see if the a priori analysis was supported by the data.

## Lesson 9

## ESTIMATED PROBABILITY

Objectives:

The child should be able to:

1. Specify the cumulative frequency of an event from the data gathered from an experiment.
2. Specify the estimated probability of an event from the data gathered from an experiment.
3. Identify that individual team estimated probabilities fluctuate to a much larger extent than the cumulative estimated probabilities.
4. Identify that for a large number of trials that the estimated probability approaches the a priori probability.
5. Specify likely bounds on the a priori probability of an event from the graph of the data of an experiment concerning the event.

Prerequisite Behaviors:

1. Collecting and organizing data.
2. Graphing and interpreting bar graphs.
3. Specifying the cumulative frequency of an event from the data.
4. An event from the data ordering of decimals of graphing their values.

Materials to be used:

1. Desk calculator.
2. Graph paper.
3. Overhead projector.
4. Transparencies.
5. Data from Lesson 8.

New Vocabulary:

None.

Method of Presentation:

Take data from tossing a coin on the board or over head and show students how to compute estimated probability as the number of favorable outcomes divided by the total number of outcomes. Have students compute estimated problem for each team. Then ask  $P(h) = ?$  Then plot a bar graph of estimated probability for each team and one for cumulative team results. Do this by having estimated team probabilities changed to decimals. Have one student punching team ratios into a calculator to change them to decimals and another student writing the decimal equivalents on the board, round off the decimals to the nearest hundreth. Plot the results on an overhead--both graphs. Compare the two graphs by having them overlap.

1. Ask: How many team results are better than the estimated probability of all the teams added together? (Only 1 out of 10--Team 3). Ask which would they rather base the estimates of the probability of heads--on 10 tosses, 40 tosses, or 400 tosses?

2. Ask: Which do you think gives the better estimate of  $P(h)$  --

one team result or all the team results added together?

Do you see that the estimated probability is slowly coming

closer to the  $P(h) = 1/2$  as the number of trials increase?

Now do the same thing with tossing a tack, only this time have them plot graphs on graph paper, while you do it one step behind them or overhead. Have one student at the board and one at the calculator to change fractions to decimals first. Ask same questions as before, only in relation to  $P(h) = ?$  (tack turns up). Ask them if they could think of something we could do to get an even better estimate of the probability of the tack turning up when it is tossed? Bring out doing an experiment or greater number of times showed give a more accurate estimate.

Do the same thing for each of the experiments as time permits.

For the ones that have not been completed, ask a student to volunteer to graph the results on an overlay.

## Lesson 9

THURSDAY, MARCH 27

Before class put picture of each experiment on board.

1. Using questions on Experiment Sheet, go over data from yesterday.
  - a. Pass out Experiment Sheets and have two people per sheet.
  - b. Put data on board for each experiment.
  - c. List only number of winners for each game.
2. Announce Quiz for Friday on Lesson 7.
3. Pass back Exercises III (Lesson 7, Part II).

## Lesson 9

## QUESTIONS FOR THURSDAY

1. What is the Probability of winning in Box A?  $P(A) =$
2. What is the Probability of winning in Box B?  $P(B) =$
3. Which game or Box would most likely make you a winner?
4. How many Winners do you expect from Box A in 50 times?
5. How many Winners do you expect from Box B in 50 times?

show data from Experiment. Write on the board.

6. Did the Box that you expected to win have the most winners?
7. Were the numbers of winners close to the expected number of winners?
8. Can you see that probability has helped us make the best decision?
9. Do you see that the Box that has the largest probability number  
also has the most Winnders in the long run?

## DISCUSSION OF LESSON 9

Thursday (3/27)

The teacher passed out the description of the experiments and had the children pair off to share the papers. The teacher then went through each experiment asking each of the ten questions for all six experiments. The pictures of the experiments had been put on the board before class. The teacher wrote the data on the board as she went over the questions. This took 35 minutes.

The teacher then talked about the diploma the masters of probability would receive and read the inscription on it. The children had different questions concerning how they earned one. They were told that if they had been masters 90 per cent or more of the time on the exercises and quizzes and if they were masters on the final test, they would receive a diploma.

At this point the observer passed out Lesson 7, Part II's exercise and announced that there would be a quiz on these concepts tomorrow. The class was instructed to correct the mistakes they had made for the remaining ten minutes of the period.

With regard to the experiments, the children were only mildly interested in considering their data from yesterday. They showed interest in the first two experiments and Experiment 5 which involved the tack and cup. Talking about these results for 35 minutes was too much.

Their comments concerning the tack and cup are of some interest. Some wanted to assign  $1/2$  to the probability of the tack turning up because there were two outcomes. Others favored  $1/2$  perhaps because of the data they had collected (29/50, 19/50, 24/50, up). The teacher referred to the model of the oddly divided spinner and said, "There are two outcomes, so is the probability here  $1/2$ ?" Most children said no. She then said, "How can you tell before you've done the experiment whether the chances on the tack are equally likely?" She mentioned that tacks are very different and some have a larger head and a short point while another has a smaller head and a longer point. The teacher asked, "Would the probability always stay the same even when the tack changes?" There was no general response. Some children weren't sure and would not respond.

The teacher then talked about the cup. She asked them how many outcomes there were when one tosses the cup. The children said, "3." When the teacher asked if they were equally likely outcomes, the students said no. The students were very definite in their preference for the tack over the cup.

The teacher made the point, "Now do you see that if you can't assign a probability number to one of two games or experiments, you have to carry out the experiment and use the data to make your judgment?"

The suggestions for change on the experiments for Lesson 8 should make the discussion of the experiments shorter and more meaningful.

## Lesson 9

FRIDAY, MARCH 29

1. Give Quiz on Lesson 7. Remind them to write their reasons why, for the second five questions.
2. Look ahead to what is going to be done.
  - a. Go over goal chart--do you know each of them?
  - b. What is to be done the rest of the period.
  - c. Monday--Review.
  - d. Tests--Tuesday and Wednesday. Start your review. Test will be on the same materials. If you get 90 per cent on this test, you will still be a Master Learner.
3. Place data on the overhead.

Coin experiment--data. Suppose we toss a coin 10,000 times and get 5,048 heads. What is the estimated probability of heads for the next throw? Mention that if one did the experiment 100,000 times, one would get a better estimate of the probability of heads.

Ask:  $P(H) = ?$

Ask: What does  $P(H) = 1/2$  mean?

4. Take data from tack experiment and place the results on the board.

Suppose you toss a tack 12,000 times, you get 8021  
and 3979

Estimated probability of                      is exactly?

Estimated probability of                      is exactly?

The chances of                      are about?

The chances of                      are about?

Suppose you have a tack with a longer point than what you've  
been throwing? What would the results be? Would it be  
the same as before?

5. Pass out Exercise 9.1--to be completed by Monday.

## DISCUSSION OF LESSON 9

Friday (3/28)

The quiz on Lesson 7 was given first. This took twenty minutes. The teacher then showed the children the goals and went over them. She told them that today they would be going over the data from their experiments and do a handout. On Monday the quiz would be discussed, the homework discussed and the rest of the time would be spent in reviewing. On Tuesday and Wednesday they would be tested.

The teacher explained that if a person was a nonmaster on today's quiz, he could become a master by doing well on the final test since there would not be enough time to give another quiz.

The teacher then showed them on the overhead the data from "tossing a coin" and talked about the data, the estimated probability of each trial, the additive total of the trials, and the estimated probability based on the additive totals.

Some children were confused on how one got .47 from  $187/400$ . The teacher divided it on the board  $400 \overline{)187}$  but some still were confused.

The teacher next put the bar graphs of the same data on the overhead. (She did not explain the scales.) The extremes in the bar graph of the estimated probability for the individual trials were compared to the small changes in the bar graph of the estimated probabilities of the additive totals.

The teacher then asked what the probability of heads =  $1/2$  means. One said that it means there is one way to win and two possible outcomes. Someone else said it means chances are when you toss it, it will come up either heads or tails and that they are equally likely. The teacher asked if it meant that if one tossed the coin six times, that one would get three heads and three tails. Some children said no, not exactly but near there. The teacher then said if you did it 10,000 times, about how many heads would you get? They said about 5,000. She then asked if it had to be exactly 5,000. The children said no.

The teacher then said that if you tossed a coin 1,000,000 times that you would come out very close to getting  $1/2$  of the tosses to be heads. When you do it ten times, you may be off by a great deal, but when you do it many times, it comes very close to  $1/2$ .

She then drew a picture of a tack and asked if she threw it whether she should expect the same results as yesterday. The children said no. The teacher said, "How would you assign a probability number to it?" One child said that you would have to do an experiment.

The imaginary data from throwing a tack was considered and the estimated probabilities of the tack turning up and down were computed. The question as to what would be about the true probability from looking at the data was considered.

The exercise for Lesson 9 was passed out as homework to be gone over on Monday.

QUIZ - LESSON 7

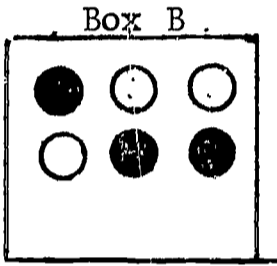
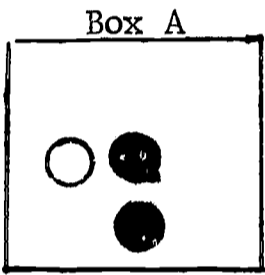
Name \_\_\_\_\_

The following boxes contain black and white marbles. To play this game you pick a marble from one of the two boxes. You win if you choose a black marble.

If you can play this game only once, do you have a better chance of winning if you pick from Box A or Box B, or doesn't it make any difference?

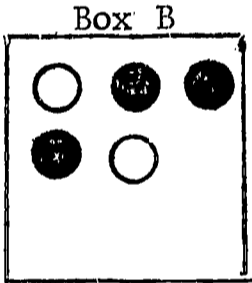
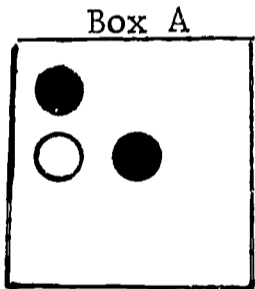
For each question, place an "X" in the blank at the right that shows your choice.

1.



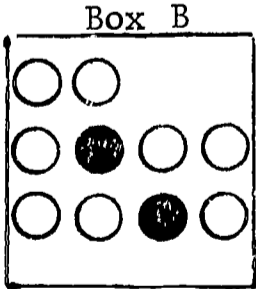
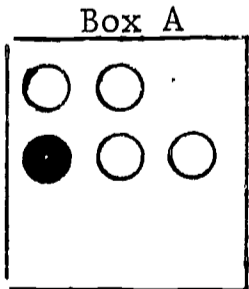
- \_\_\_\_\_ Box A  
\_\_\_\_\_ Box B  
\_\_\_\_\_ It doesn't make any difference

2.



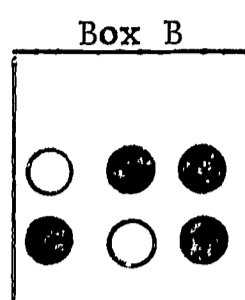
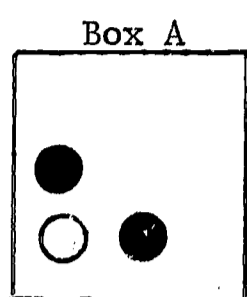
- \_\_\_\_\_ Box A  
\_\_\_\_\_ Box B  
\_\_\_\_\_ It doesn't make any difference

3.



- \_\_\_\_\_ Box A  
\_\_\_\_\_ Box B  
\_\_\_\_\_ It doesn't make any difference

4.

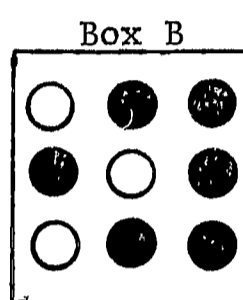
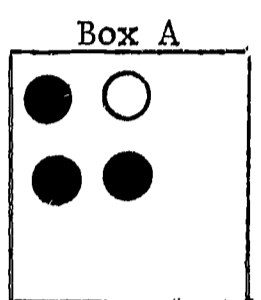


\_\_\_\_\_ Box A

\_\_\_\_\_ Box B

\_\_\_\_\_ It doesn't make any difference

5.



\_\_\_\_\_ Box A

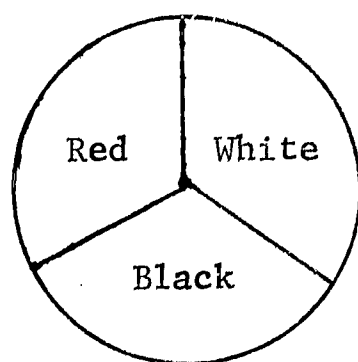
\_\_\_\_\_ Box B

\_\_\_\_\_ It doesn't make any difference

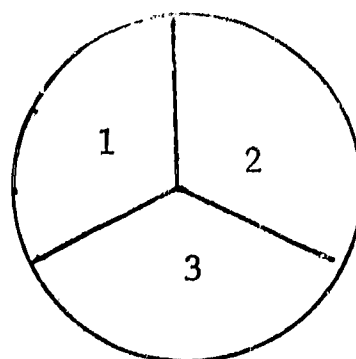
The following set of problems ask you to choose the game which gives you the best chance of winning. If the chances are the same you are to identify that it doesn't make any difference. Place an X in the blank to the left of the choice which you think is correct. Give a short explanation as to why you gave the answer you did. If you do not know the answer to a problem, leave it blank. Do not guess.

1. Which game would you rather play or doesn't it make any difference?

First Spinner



Second Spinner



\_\_\_\_\_ Game 1 You spin the first spinner once and you win if you get red.

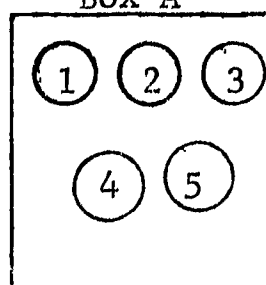
\_\_\_\_\_ Game 2 You spin both spinners once and you win if you get red and a "1" or red and "2".

\_\_\_\_\_ It doesn't make any difference.

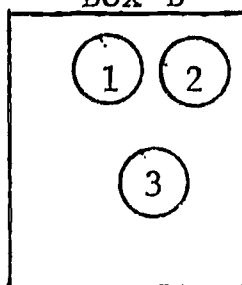
Why?

2. Which game would you rather play or doesn't it make any difference?

Box A



Box B



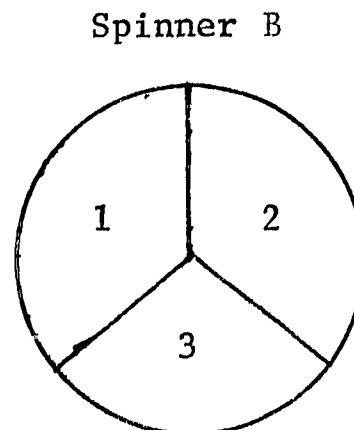
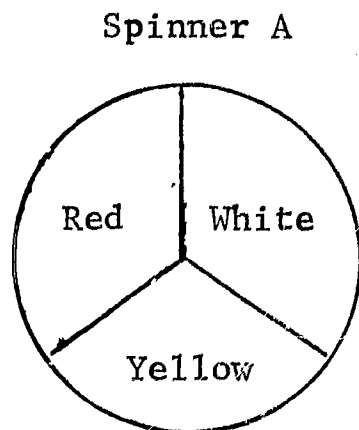
\_\_\_\_\_ Game 1 You pick a chip from box A and you win if you pick the chip with "3" on it.

\_\_\_\_\_ Game 2 You pick one chip from Box A and one chip from Box B. You win if you get a sum of "4".

\_\_\_\_\_ It doesn't make any difference.

Why?

3. Which game would you rather play or doesn't it make any difference?



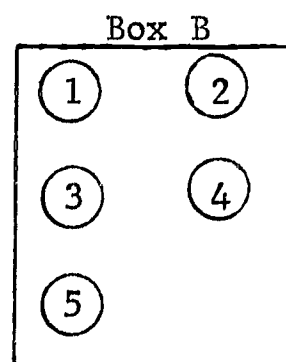
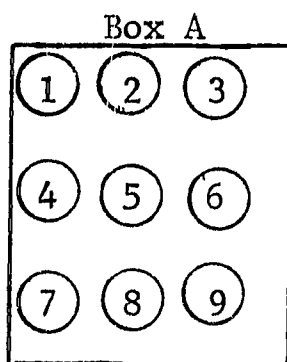
\_\_\_\_\_ Game 1 You spin Spinner A once and you win if you get white.

\_\_\_\_\_ Game 2 You spin Spinner B twice. You win if you get a sum of 3.

\_\_\_\_\_ It doesn't make any difference.

Why?

4. Which game would you rather play or doesn't it make any difference?



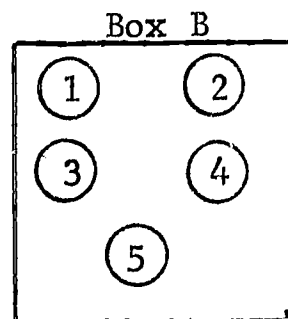
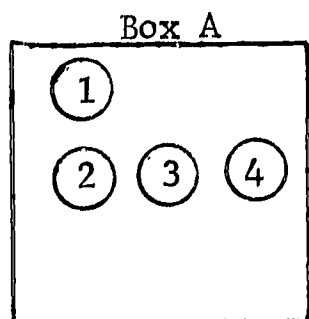
\_\_\_\_\_ Game 1 You pick a chip from Box A and you win if you pick the chip with "3" on it.

\_\_\_\_\_ Game 2 You pick twice from Box B without putting the first chip you pick back. You win if you pick a "2" first and a "3" on the second pick.

\_\_\_\_\_ It doesn't make any difference.

Why?

5. Which game would you rather play or doesn't it make any difference?



\_\_\_\_\_ Game 1    You pick one chip from Box A and you win if you get a chip with a "2" on it.

\_\_\_\_\_ Game 2    You pick one chip from Box A and one from Box B and find the sum of the numbers on the chips. You win if you get a sum of "3" or "6".

\_\_\_\_\_ It doesn't make any difference.

Why?

### ANALYSIS OF LESSON 9 QUIZ

The mean on the ten item quiz was 8.88 or 88.8 per cent. The variance was 2.35. 19/25 had 90 per cent or better; 21/25 had 80 per cent or better; 22/25 had 70 per cent or better. The three students scoring below 70 per cent were subjects 10, 15 and 20.

#### The Item Difficulties Were

<u>1-D</u>	<u>1-D and 2-D</u>
1. 23/25	1. 23/25
2. 23/25	2. 22/25
3. 25/25	3. 21/25
4. 23/25	4. 21/25
5. 22/25	5. 18/25

Item 5, page 2 was the most difficult item. 4/7 incorrect responses had assigned the correct probabilities but had either ordered the fractions incorrectly or made the wrong decision. Part of this could be due to the subjects' unfamiliarity with fractions expressed in twentieths.

Subject 10 appeared to have difficulty in ordering two fractions.

Subjects 15 and 20 have been nonmasters on practically every exercise and quiz. They need more help on assigning probabilities, ordering ratios and making a decision on the basis of the ordered fractions.

Lesson 12 - Ghosts, Goblins and "Coins that Remember".

1. Do you believe that there are ghosts?
2. Do you believe that there are goblins?
3. Do you believe that a coin can remember?

You probably answered "No" to all these questions. Yet often we hear people talk as if they believe that coins can think and remember. They really do not understand the ideas in the law of large numbers. Most people call it the law of averages, and they often draw wrong conclusions from it.

You have heard people say:

- A. "I have tossed an honest coin four times. Each time it came up heads. The law of averages says that the next toss will be tails."

Do you believe that the next toss is more likely to be tails than heads?

- B. "My teacher uses a spinner to assign positions for the baseball game of "work up". I haven't been assigned as a pitcher yet this year. Therefore, by the law of averages, I'm sure to be assigned as pitcher today."

Do you think that this pupil is more likely than not to be chosen as a pitcher?

- C. "I have been tossing an honest die. In 23 tosses, the face with one dot on it has never been up. By the law of averages, it is very likely that it will come up on the next toss."

Do you think the face with one dot is more likely than any other face?

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\*Taken from SMSG's Probability for Intermediate Grades, pp. 80-87.

Let's look at each of these examples of a misunderstanding of the "law of averages". Look back at statement A.

- A. A coin does not have a memory. It cannot "remember" that it has been heads on the last four tosses. There is an equal chance for heads or for tails on the next toss.

We can use mathematics to prove that it is "unusual" to have a coin show four heads in four tosses. We can draw a tree diagram<sup>or</sup> make a table.

How many different outcomes are there when 4 coins are tossed or when one coin is tossed 4 times? How many of these outcomes consist of four heads? So,  

$$P(4 \text{ heads}) = \frac{1}{16}.$$

However, this also means that we expect 4 heads in a row, once every 16 times that we toss 4 coins. The coin while flying through the air on the fifth toss cannot say to itself, "Well, that's 4 heads in a row; I better twist a bit more and be sure to land tails or I'll mess up the 'law of averages'." The probability of heads on the next toss is of course  $\frac{1}{2}$ , the same as any other individual toss. Some people who misunderstand the law of averages think the probability of tails is much greater than  $\frac{1}{2}$  after a coin has been heads several times in a row. Do you know people like this? They have forgotten that what happens on one toss has NOTHING to do with what will happen on the next toss.

Refer to statements B and C.

- B. If there are 9 positions on the baseball field, then the probability of getting any one position is 1 out of 9. The fact that this pupil has not been a pitcher yet does not cause the spinner to favor one position for him over the others. He still has only 1 chance in 9 of being a pitcher today.
- C. This person is overlooking one simple fact about a die -- it cannot think! It cannot say, "Let's see now. I know the probability of any face is  $\frac{1}{6}$ . My face with one dot on it has not been up in 23 tosses, so on the next toss I'll land so that the face with the one dot is on the top."

This person is thinking, "One face hasn't been up for a long time, so that face is more likely to come up than any of the others." This is a mistake about the law of averages that people often make. He doesn't really believe that dice can think, yet he is acting as if they could. Each face on a die has just as much chance to be up as any other face. If the face with one dot has not been up in 100 tosses, it still has no more chance than any other face to be up on the next toss. In fact, it has just one chance out of six.

Can you think of other correct or incorrect statements that you have heard about the law of averages? List some of them.

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Why do so many people misunderstand the law of averages? It is too bad, but we all believe things and arrive at conclusions which just aren't true.

Which of these statements are false?

1. Lightning never strikes twice in the same place.
2. If you handle a frog, you'll get warts.
3. The end of the Panama Canal on the Pacific Ocean side is farther west than the end on the Atlantic Ocean side.
4. Horses are smarter than pigs.
5. George Washington threw a dollar across the Potomac River.
6. Columbus discovered America.

Many people believe some of these statements. Did you believe any of them? If you did, it isn't at all surprising. However, the six statements are all false. Most of us believe some things which really aren't true. Why is this so? There are many reasons. Among them are:

1. We are told or we have read something which is not true, but we remember it.
2. We did not understand what we were told or what we have read.
3. We reasoned incorrectly.
4. Our experience caused us to believe something that wasn't true.

5. We jumped to a conclusion without knowing enough facts.
6. We failed to check our belief against the facts.

This list could go on and on. There may be other reasons that you can think of. People arrive at false ideas about the law of averages for many of these same reasons. We can be fooled unless we are very careful. We might believe that an outcome, such as heads on a toss of a coin, is bound to happen if it hasn't happened for many tosses. It is easy to understand how our brain fools us in this case. It tells us that for a large number of tosses of a coin, heads will occur about half of the time -- and this is true. This is an example of the law of large numbers. Then we observe that heads hasn't occurred for several tosses and we make the mistake of thinking that heads must now start occurring more often to "catch up" with the number of tails. This is not true. Remember, a coin can't think. On each toss, there is just as much chance for heads to turn up as for tails.

By using mathematics, we can learn many interesting things. For example, from 15 children in your room, there are 6,435 different ways you can have 7 children on a committee. If you choose a 7-member committee from 30 students, you have a choice of 2,035,800 different committees. Another example is if a coin has been tossed and heads have occurred 7 out of 10 times, chances are less than  $\frac{1}{2}$  that tails will "catch up" in 100 tosses. The mathematician can tell what will probably happen in cases such as this.

The next time that you hear some statement about the "law of averages", listen carefully. Try to find what the person believes and see if he is using it correctly.

Exercises - Lesson 12.

Mark these TRUE or FALSE .

- \_\_\_\_\_ 1. You have been spinning a spinner that has a dial which is  $\frac{1}{2}$  black and  $\frac{1}{2}$  red. The last four spins have landed on black. It is more likely that the spinner will show red on the next spin than black.
- \_\_\_\_\_ 2. The last five new pupils who came to our school were boys. The chances are better than equal that the next new pupil will be a girl.
- \_\_\_\_\_ 3. The hospital reported that the last seven babies born there were girls. It is more likely that the next baby born there will be a boy than that it will be a girl.
- \_\_\_\_\_ 4. The weatherman says that on the average it rains 4 days during the month of July. Today is the 27th of July and it has not rained all month. Therefore, it will rain tomorrow.
- \_\_\_\_\_ 5. An auto dealer has 250 new cars and he knows that one out of every five new cars he sells is colored black. This week he has sold a blue, a white, a green, and a grey car. It is more likely than not that the next car he sells will be a black one.

## Things To Do At Home - Lesson 12.

This experiment may help you to see why some people draw wrong conclusions from the law of averages.

### Problem:

How many times, on the average, do you think that you would have to toss a coin before it comes up heads? Use the chart on the next page.

### Procedure:

Toss a coin. Count the number of tosses until you get a head. For example: If you get a head on the first toss, write 1 in the column just to the right of "1st head". Start over. If you do not get a head until the fourth toss, write a 4 just to the right of "2nd head". Continue until you have completed column A. Repeat for columns B through E. Each column provides spaces to record the tosses for 10 heads.

After you have tossed 50 heads, add the number of tosses to get each group of ten heads. Divide each of these sums by 10 to find the average number of tosses needed to get one head.

Then, add the sums from the five columns and divide by 50. This gives the average number of tosses to get one head. Is this average closer to 2 than the average for each of the five columns? How many times did it take more than 5 tosses to get a head? How many times did it take 2 tosses to get a head? How many times did it take only 1 toss to get a head?

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## Number of Tosses to Get a Head

1st head					
2nd head					
3rd head					
4th head					
5th head					
6th head					
7th head					
8th head					
9th head					
10th head					
SUM					
$SUM \div 10$					
	A	B	C	D	E

Perhaps now you can see why some people misunderstand the law of averages. With many tosses of a coin, we do see that about half of the tosses are heads. That is, it takes 2 tosses, on the average, to get heads. But, when you tossed a coin, you found that sometimes you tossed a head on only 1 toss. Other times, you had to toss the coin several times to get a head. This should help you understand that these people fail to see that the "average" is made from numbers that differ quite widely and that there is NOT a "law" which says that you must get a head after tossing 5 tails, for example.

## REVIEW LESSON

Monday (3/31)

1. Hand Quiz Lesson 7 back, discuss weaknesses and mistakes.
2. Homework back, discuss mistakes on Lesson 9.
3. Review

- a. Dice--Die

Starting with a die and then dice, ask for outcomes and

and write down the answers. ( $P(3) = ?$ ;  $P(\text{sum} = 3) = ?$ )

( $P(\text{even number})$ ;  $P(2 \text{ even numbers})$ ).

- b. Estimated Probability

Use data from cup.

Put data on board.

Talk about estimated probability for results.

Outcomes--are they equally likely?

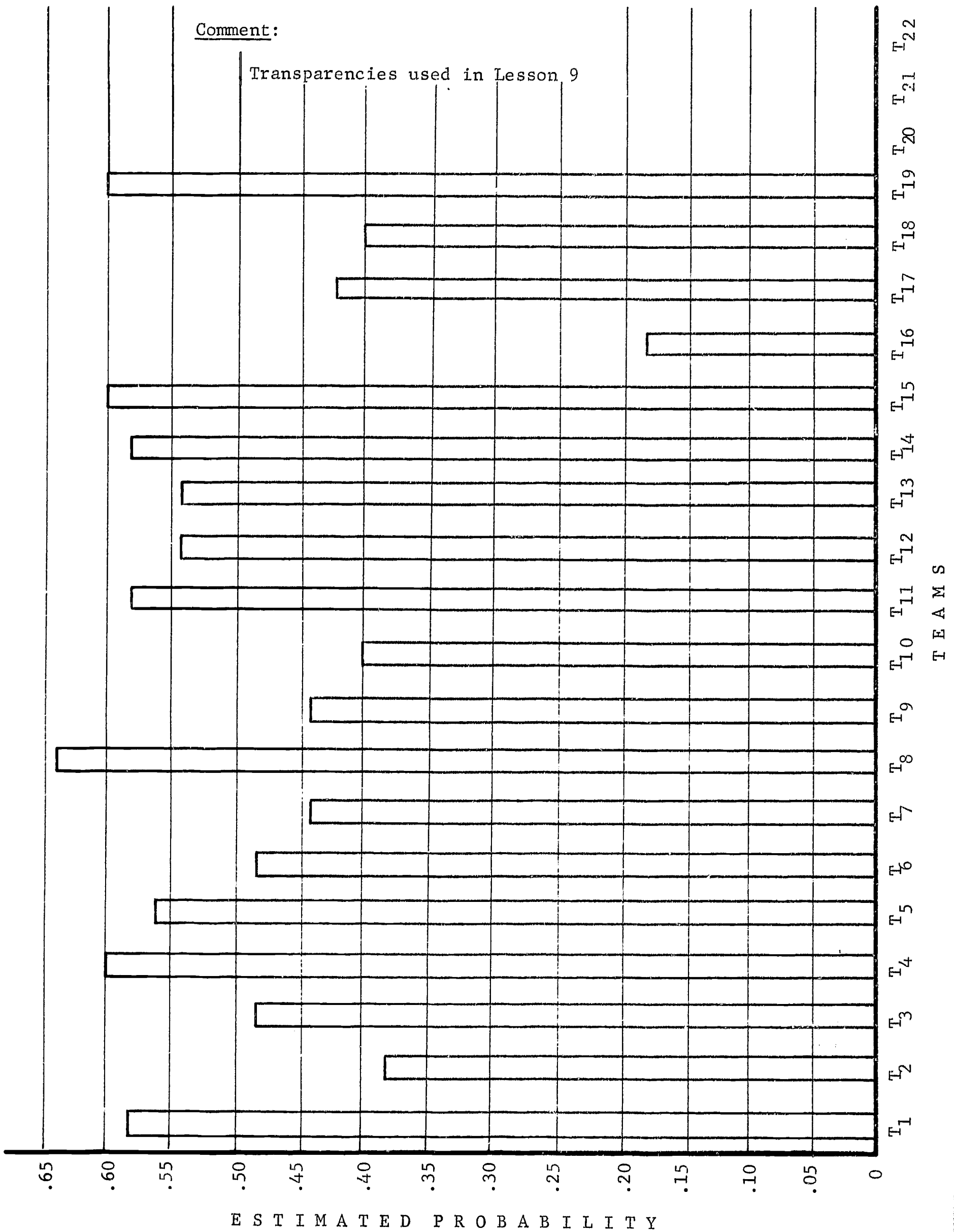
- c. Review on own by going over papers

Masters on their own.

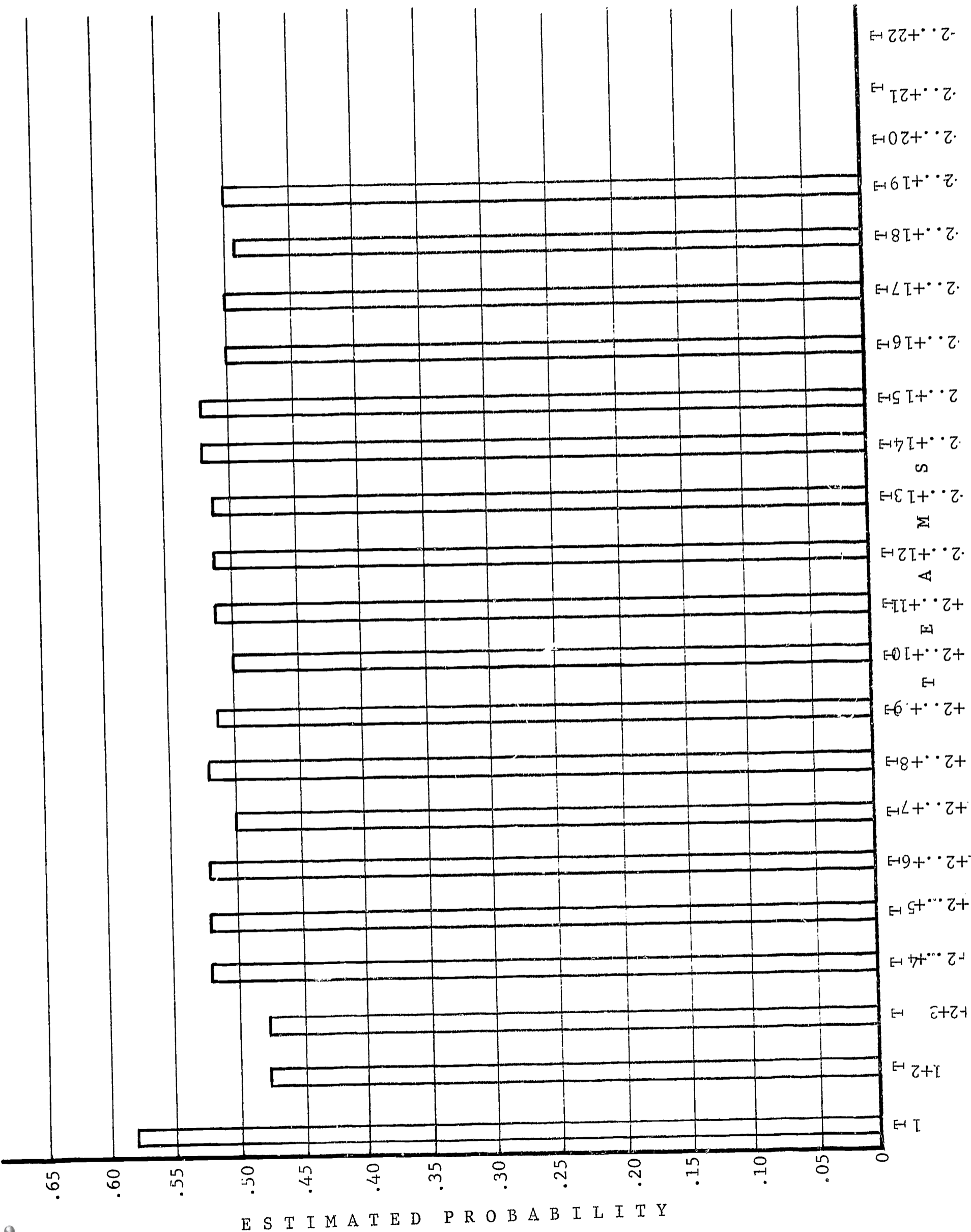
Nonmasters (on quiz)--we will help.

- d. Review goals from goal chart at end.

## TOSSING A TACK (INDIVIDUAL TRIALS)



Transparency: Lesson 9  
TOSSING A TACK (ADDITIVE TRIALS)



MONDAY 3/31

Monday's lesson began by the teacher returning and discussing the quiz from Friday. Three problems (p. 1 #2, p. 4 #4, p. 5 #5) were discussed in detail with the children drawing the trees on the board for #4 and #5.

The homework was then discussed.

The teacher then reviewed the die and dice models, asking questions concerning outcomes and probabilities.

The data from the children's throwing a cup, a tack, and spinning the two spinners were placed on the overhead and discussed. The range of the estimated probability for the individual trials was compared to the range of the estimated probability of the additive trials.

For example, from the data on the cup, the range of the individual trials of .00 to .16 for the cup turning up was brought out. The teacher then asked if the children would assign a probability number from these results. There was no response. The teacher then looked at the additive total and the ratios. She repeated the question and still got no response. She then brought out that the data supported assigning the probability of about 7 per cent to the chances of the cup turning up. The fact that the three outcomes of tossing the cup were unequally likely was recognized by the students. When the teacher asked why, a girl replied that the cup had more area on its side and will land on the side more often and thus the outcomes aren't equally unlikely.

After the discussion of the data from the models, the class was instructed to begin their review for tomorrow's quiz by looking over their old papers and seeing their mistakes. In response to questions about tomorrow's test, the teacher told the children that there wouldn't be any type of question on the test that hadn't already been done.

At the end of the period the teacher briefly went over each goal from the goal chart.

After the period, the teacher and the author worked with subjects 15 and 20. From an analysis of their mistakes in quiz on Lesson 7, it was found that subject 15 was not using the probability numbers in making a decision and that subject 20 was assigning the probabilities to the events (finally) but was only using the numerator to make her decision (regardless of the denominator). Individual tutoring was given them on the basis of this analysis for approximately 30 minutes

## Multiple Choice -

Identify whether the following fractions are equal or if one is larger than the other.

\_\_\_\_\_ 1.  $1/4$ ,  $2/6$

- (a)  $1/4 = 2/6$                       (b)  $1/4$  is greater than  $2/6$   
(c)  $2/6$  is greater than  $1/4$

\_\_\_\_\_ 2.  $1/2$ ,  $2/4$

- (a)  $1/2 = 2/4$                       (b)  $1/2$  is greater than  $2/4$   
(c)  $2/4$  is greater than  $1/2$

\_\_\_\_\_ 3.  $2/10$ ,  $3/9$

- (a)  $2/10 = 3/9$                       (b)  $2/10$  is greater than  $3/9$   
(c)  $3/9$  is greater than  $2/10$

\_\_\_\_\_ 4.  $2/3$ ,  $3/6$

- (a)  $2/3 = 3/6$                       (b)  $2/3$  is greater than  $3/6$   
(c)  $3/6$  is greater than  $2/3$

\_\_\_\_\_ 5.  $1/2$ ,  $3/7$

- (a)  $1/2 = 3/7$                       (b)  $1/2$  is greater than  $3/7$   
(c)  $3/7$  is greater than  $1/2$

\_\_\_\_\_ 6.  $2/3$ ,  $3/5$

- (a)  $2/3 = 3/5$                       (b)  $2/3$  is greater than  $3/5$   
(c)  $3/5$  is greater than  $2/3$

\_\_\_\_\_ 7.  $1/5$ ,  $2/10$

- (a)  $1/5 = 2/10$                       (b)  $1/5$  is greater than  $2/10$   
(c)  $2/10$  is greater than  $1/5$

\_\_\_\_\_ 8.  $2/10$ ,  $2/9$

- (a)  $2/10 = 2/9$                       (b)  $2/10$  is greater than  $2/9$   
(c)  $2/9$  is greater than  $2/10$

Comment: This exercise was only given to five students (subjects 1, 10, 15, 20 and 24) who needed practice in ordering two fractions.

\_\_\_\_\_ 9.  $3/8$ ,  $2/5$

(a)  $3/8 = 2/5$  (b)  $3/8$  is greater than  $2/5$

(c)  $2/5$  is greater than  $3/8$

\_\_\_\_\_ 10.  $2/3$ ,  $4/6$

(a)  $2/3 = 4/6$  (b)  $2/3$  is greater than  $4/6$

(c)  $4/6$  is greater than  $2/3$

\_\_\_\_\_ 11.  $3/5$ ,  $6/10$

(a)  $3/5 = 6/10$  (b)  $3/5$  is greater than  $6/10$

(c)  $6/10$  is greater than  $3/5$

\_\_\_\_\_ 12.  $2/10$ ,  $1/4$

(a)  $2/10 = 1/4$  (b)  $2/10$  is greater than  $1/4$

(c)  $1/4$  is greater than  $2/10$

\_\_\_\_\_ 13.  $2/4$ ,  $4/8$

(a)  $2/4 = 4/8$  (b)  $2/4$  is greater than  $4/8$

(c)  $4/8$  is greater than  $2/4$

\_\_\_\_\_ 14.  $2/5$ ,  $1/2$

(a)  $2/5 = 1/2$  (b)  $2/5$  is greater than  $1/2$

(c)  $1/2$  is greater than  $2/5$

\_\_\_\_\_ 15.  $3/10$ ,  $2/6$

(a)  $3/10 = 2/6$  (b)  $3/10$  is greater than  $2/6$

(c)  $2/6$  is greater than  $3/10$

\_\_\_\_\_ 16.  $3/6$ ,  $2/4$

(a)  $3/6 = 2/4$  (b)  $3/6$  is greater than  $2/4$

(c)  $2/4$  is greater than  $3/6$

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- \_\_\_\_\_ 17.  $3/4$ ,  $4/6$   
(a)  $3/4 = 4/6$  (b)  $3/4$  is greater than  $4/6$   
(c)  $4/6$  is greater than  $3/4$
- \_\_\_\_\_ 18.  $1/3$ ,  $2/6$   
(a)  $1/3 = 2/6$  (b)  $1/3$  is greater than  $2/6$   
(c)  $2/6$  is greater than  $1/3$
- \_\_\_\_\_ 19.  $1/5$ ,  $1/6$   
(a)  $1/5 = 1/6$  (b)  $1/5$  is greater than  $1/6$   
(c)  $1/6$  is greater than  $1/5$
- \_\_\_\_\_ 20.  $1/3$ ,  $1/6$   
(a)  $1/3 = 1/6$  (b)  $1/3$  is greater than  $1/6$   
(c)  $1/6$  is greater than  $1/3$
- \_\_\_\_\_ 21.  $1/3$ ,  $2/9$   
(a)  $1/3 = 2/9$  (b)  $1/3$  is greater than  $2/9$   
(c)  $2/9$  is greater than  $1/3$
- \_\_\_\_\_ 22.  $1/3$ ,  $1/9$   
(a)  $1/3 = 1/9$  (b)  $1/3$  is greater than  $1/9$   
(c)  $1/9$  is greater than  $1/3$
- \_\_\_\_\_ 23.  $2/8$ ,  $3/9$   
(a)  $2/8 = 3/9$  (b)  $2/8$  is greater than  $3/9$   
(c)  $3/9$  is greater than  $2/8$
- \_\_\_\_\_ 24.  $2/10$ ,  $1/8$   
(a)  $2/10 = 1/8$  (b)  $2/10$  is greater than  $1/8$   
(c)  $1/8$  is greater than  $2/10$

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- \_\_\_\_\_ 25.  $2/3, 2/4$   
(a)  $2/3 = 2/4$       (b)  $2/3$  is greater than  $2/4$   
(c)  $2/4$  is greater than  $2/3$
- \_\_\_\_\_ 26.  $3/8, 2/5$   
(a)  $3/8 = 2/5$       (b)  $3/8$  is greater than  $2/5$   
(c)  $2/5$  is greater than  $3/8$
- \_\_\_\_\_ 27.  $1/4, 7/16$   
(a)  $1/4 = 7/16$       (b)  $1/4$  is greater than  $7/16$   
(c)  $7/16$  is greater than  $1/4$
- \_\_\_\_\_ 28.  $1/2, 2/6$   
(a)  $1/2 = 2/6$       (b)  $1/2$  is greater than  $2/6$   
(c)  $2/6$  is greater than  $1/2$
- \_\_\_\_\_ 29.  $1/3, 1/4$   
(a)  $1/3 = 1/4$       (b)  $1/3$  is greater than  $1/4$   
(c)  $1/4$  is greater than  $1/3$
- \_\_\_\_\_ 30.  $1/5, 6/20$   
(a)  $1/5 = 6/20$       (b)  $1/5$  is greater than  $6/20$   
(c)  $6/20$  is greater than  $1/5$

## POSTTESTING

The posttest was administered on Tuesday (4/1) and Wednesday (4/2). Part I, given on Tuesday, took 35 minutes. Part II, given on Wednesday, took 25 minutes.

After the children had completed Part I, the children asked two questions:

1. Was the answer to the dice problem ((R 12-1), "How many ways can all the sums be gotten?" (when two dice are thrown)) 36? The investigator hesitated in answering the question. But after one of the brightest boys in the class said yes, the investigator confirmed his reply. A cheer went up from the majority of the class. Some of the best students had asked about the question or a parallel question that used the same wording during the test. The investigator did not answer their questions at that time.
2. What is the answer for the question about throwing a tack (R - 14 - 1)? The investigator discussed the item, but the students seemed confused concerning the wording of the item.

On Wednesday, after administering Part II the students asked more questions. The one they were most concerned with was R - 13 - 1 on the estimated probability. The investigator also discussed the solution of the two brainteasers given earlier concerning the total number of

telephone numbers possible using seven digits and the probability of having children in a family in a particular order (e.g.,  $P(\text{Girl, Boy, Girl}) = 1/8$ .) The investigator tried to get them to generalize from the tree to use the product rule. However, many of the group did not seem ready to do this yet.

#### Awarding of Diplomas:

The diplomas were presented to the group on April 11 by the investigator and the teacher used in the study. All but two received their masters diplomas. Subjects 15 and 20 were given mastery diplomas at the B level of work.

The investigator discussed the items on the posttest concerned with the question, "How many ways can you get . . .?" and Item R - 13- 1 on the approximate probability.

Unfortunately, there was no blackboard to write on. Thus the discussion was purely a verbal discussion of the previously mentioned problems.